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Non-Conservative Gravitational Fields Around Spinning Spherical Objects (and the Absence of Gravitomagnetic Fields) as Revealed by the Kerr Metric

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Abstract

The Kerr metric in the equatorial plane of a spinning, spherical mass is scrutinized. It turns out that the gravitational field lines of a spinning, massive sphere are not strictly straight, but are, in the equatorial plane, curved because of a tangential component. This was mentioned by N.A. Sharp in 1979, although without any proof or reference. An equation for determining the tangential component of the gravitational field is provided. Thereby it is shown that a gravitational Lorentz force and hence a gravitomagnetic field do not exist, although this had been postulated by Heaviside and Thirring. Because of the tangential component, the gravitational field around a spinning spherical body is not conservative. Hence, there is an analogy to the electric field, which, too, can either be conservative (as is the case for the electrostatic field), or non-conservative (which is the case whenever the magnetic flux encircled by a path in space is subject to change). An orbiter held on a circular trajectory in the equatorial plane and circling with the spin of the central mass thus experiences a steady onward force like a charged particle in a ring-shaped particle accelerator does. The gain in kinetic energy of an orbiter is at the expense of the rotational kinetic energy of the central, spinning mass. This is because the gravitational field lines

of any orbiter, too, are curved, and thus exert a torque on the central, spinning mass.

Keywords: General Relativity, Kerr metric, gravitational field, geodesics, M 87 jet, spinning black hole, Lorentz invariant, time reversal

Introduction

aa) The Kerr metric is an exact solution of Einstein’s field equation for spinning spherical bodies. Before this solution of Einstein’s field equation was found in 1963, it had been speculated that a fast spin of a massive object would create a second field analogous to what happens when an electrically charged sphere is spinning. In the latter case, a magnetic field is generated. It was O. Heaviside (Heaviside 1893) who first postulated such an extension of gravity. He did that as early as 1893, but has been rarely mentioned in the context of “gravitomagnetism”. A field analogous to the magnetic field has thus been speculated to exist in the gravitational case (“gravitomagnetic” field). As a consequence of this, it has been believed that a particle on an orbit around a spinning stellar object in its equatorial plane is forced to move in cycloids along the orbital trajectory. This, in turn, has often been illustrated by imagining that space itself is dragged along with the spinning spherical body like a viscous liquid.

The cycloidal motion is thought to be caused by a combination of a “gravitomagnetic field”, a “gravitomagnetic Lorentz force” experienced by a massive test body when moving in the “gravitomagnetic field”, and the ordinary gravitational field. The co-existence of the ordinary gravitational field, the “gravitomagnetic field” and the “gravitational Lorentz force” would result in a phenomenon whose counterpart in electromagnetism is called $\mathbf{E} \times \mathbf{B}$ -drift. It is that drift which makes charged particles move in cycloids in the electromagnetic case.

The mechanism behind the drift is, however, is (undoubtedly) incapable of accelerating an object. For a Lorentz force is categorically incapable of doing work, simply because the force on a moving object and the direction of motion form a right angle. Seeing the “gravitational Lorentz force” as the cause of jets near black holes has therefore been out of the question.

bb) The postulated cycloidal motion of test bodies on an orbit in the equatorial plane is often said to be caused by a laminar flow of space dragged along by the rotating central mass. Although a flow of space does indeed exist (as it turns out to be a consequence of the Kerr metric), the common picture of what goes on in the vicinity of a fast spinning spherical object is nevertheless misleading: It does not give expression to the fact that the gravitational field around a fast spinning object has, according to the Kerr

metric (which replaces all previous speculations), a tangential component (in addition to its anti-radial component), as will be proved. An object on a circular trajectory around the spinning body is thus subject to a permanent acceleration in the tangential direction, similar to what happens to a charged particle in a ring-shaped particle accelerator.

cc) Because of its tangential component, the gravitational field in the vicinity of a spinning spherical body is non-conservative. A corrected conception of flowing space (instead of a liquid swirling around the spinning object) which brings this property to light is provided when the flow of space is recognized to behave analogously to what it undisputedly does in the *cosmological* variant of the Schwarzschild solution (expanding De-Sitter-universe). This is to say:

– As is well known, the cosmic version of the Schwarzschild equation (which is the solution of Einstein’s field equation for a spherical, non-spinning mass) is arrived at by setting the density of mass or energy, that is the tensor component \mathbf{T}^{00} , to zero in Einstein’s field equation (all the other components of \mathbf{T} are zero anyway), and Einstein’s cosmological constant on the left-hand side to a positive, non-zero value. More precisely: The Schwarzschild solution for any spherically symmetric arrangement is (given spatial displacements shall occur in the radial direction only): $d\mathbf{tau}^2 = \mathbf{f}(\mathbf{r})d\mathbf{t}^2 - d\mathbf{r}^2/\mathbf{f}(\mathbf{r})$, with $\mathbf{f}(\mathbf{r})$ either being equal to $\mathbf{1}-2\mathbf{a}/\mathbf{r}$ or to $\mathbf{1}-\mathbf{b}\mathbf{r}^2$.

– For the neighborhood of a spherical mass, the first alternative is used, and the constant \mathbf{a} is set equal to $\mathbf{GM}/\mathbf{c}^2 = \mathbf{r}_s/2$ (with \mathbf{G} being Newton’s gravitational constant, \mathbf{r} being radial distance, i.e., circumference of a circle with the spherical mass at its center, divided by 2π , \mathbf{tau} being proper time, \mathbf{M} being the mass of the spherical object, \mathbf{r}_s being the Schwarzschild radius, \mathbf{c} being the speed of light).

– For a de-Sitter universe (of the kind in which dark energy prevails over all other forms of energy), the second alternative is used, with the constant \mathbf{b} being set equal to $\mathbf{H}^2/\mathbf{c}^2$ (where \mathbf{H} is Hubble’s constant, that is the increase in the galaxies’ escape velocity with distance from the Milky Way). The numerical value of Einstein’s cosmological constant determines the numerical value of Hubble’s constant.

– In an expanding universe like the one described by the cosmic variant of the Schwarzschild solution, space volume elements emerge out of nothingness everywhere, in order to flow in a straight, but *accelerating* motion toward the event horizon, where the escape velocity is \mathbf{c} (for a local observer who is stationary relative to the Milky Way galaxy).

– In the variant for a black hole (where the focus of our attention is), emerging space volume elements flow towards the Schwarzschild horizon in an accelerating motion (as will be shown), where their velocity reaches \mathbf{c} (for an observer at rest in front of the Schwarzschild horizon). This parallel between the cosmic situation and black holes has so far not been realized in the literature.

In case the black hole is spinning, the accelerating flow is no longer strictly straight (strictly anti-radial), but has a tangential component, too.

dd) The tangential component of this flow of space (which goes along with a tangential component of the gravitational field) can – in addition to the Penrose process – provide the mechanism of creating enormous jets which originate in the immediate vicinity of super-massive black holes. Since a non-conservative gravitational field around the spinning spherical body is a consequence of the Kerr metric, which, in turn, is a solution of Einstein’s field equation, the apparently strange phenomenon of a “kinetic energy production apparently out of nothingness” in these fields (see below for a correction of this first, wrong impression) could be reproduced by any computer simulation of jets near spinning black holes. A requirement of such a simulation would be to set the covariant divergence of the tensor \mathbf{T} appearing on the right-hand side of Einstein’s field equation to zero (see below). It seems that this has already been achieved in a simulation by J. Davelaar, H. Olivares, O. Porth, et al. (Davelaar, Olivares, Porth, et al. 2019)(Section 2.1, Eq. 1b), and in another one by A. Cruz–Osorio, Ch. M. Fromm, Y. Mizuno, et al. (Cruz-Osorio, Fromm, Mizuno, et al. 2021).

ee) The first impression of a creation of kinetic energy “out of nothingness” (that might arise) is wrong: The rotational energy of the spinning black hole can be assumed to be reduced to the same extent to which the tangential component of its non-conservative gravitational field generates kinetic energy of accelerated particles (or larger objects) in the vicinity of the spinning mass (see Cruz–Osorio, Fromm, Mizuno, et al. 2021: “... *the spin of the BH being a potential source of energy for the launching mechanism, ...*”). The gravitational field lines generated by objects orbiting the fast spinning mass (black hole), too, are not straight, but are slightly curved due to the distortion of spacetime caused by the central, spinning mass. Even if the density of the central mass is radial symmetrical in a horizontal plane (different from the earth-moon system where the tidal bulge and the time span needed for relaxation is crucial for explaining the slowing down of Earth’s spinning rate), the orbiters’ field lines, for the reason of being curved, nevertheless result in a torque exerted on that central, spinning mass.

This is quite different from what has so far been taken into account as a mechanism of extracting rotational energy from the spinning mass by most authors. Recently, D. Garofalo and Ch. B. Singh (Garofalo, Singh 2021) re-considered possible *electromagnetic* processes which had been suggested (and criticized), and arrived at the conclusion that some central questions remain unanswered.

1. Proof of the presence of a tangential component of the gravitational field around a spinning spherical body

by showing that a strictly radial trajectory cannot be a geodesic

a) The equation of a geodesic

aa) In most of the earlier literature on geodesics in the Kerr metric, it was tacitly but compellingly presumed (for the purpose of applying the method of separation of variables in the Hamilton-Jacobi equation) that a test particle in free anti-radial fall in the equatorial plane (coming from far away) would not acquire any tangential component of motion. See for instance J.M. Bardeen (p. 220):

“The symmetries of the Kerr metric give immediately two constants of the motion along the trajectory, the energy relative to infinity ... and the angular momentum about the symmetry axis These are sufficient to determine trajectories in the equatorial plane,”

But given \mathbf{m}_1 is the mass of the gravitating object and \mathbf{m}_2 is the mass of the test body, the centrifugal forces felt by \mathbf{m}_1 on the one hand and by \mathbf{m}_2 on the other hand (when the two objects rotate about an axis that passes through the center of mass of the system) must be equal in absolute amount. In simple mathematical terms:

$$m_1 r_1 \omega^2 = m_2 r_2 \omega^2 \quad \Leftrightarrow \quad \frac{r_1}{r_2} = \frac{m_2}{m_1}$$

The quotient of the two angular momenta \mathbf{p}_1 and \mathbf{p}_2 (the sum of which is conserved during any interactions between the two objects) thus amounts to:

$$\frac{p_1}{p_2} = \frac{m_1 r_1^2 \omega}{m_2 r_2^2 \omega} = \frac{m_1 m_2^2}{m_2 m_1^2} = \frac{m_2}{m_1}$$

The quotient vanishes if \mathbf{m}_1 is very large compared to \mathbf{m}_2 . Consequently, in such a case the whole momentum of the system is carried by the test object \mathbf{m}_2 . This is true for classical mechanics, and there is no reason why it should not be true in Relativity as well. Given the total angular momentum of the system is conserved, the angular momentum of the small test object \mathbf{m}_2 , too, is thus conserved, just as presumed by Bardeen. It stays the same even if its trajectory is not a perfect circle, but an ellipse.

However, this is true only as long as the test object is not subject to a tangential force (tangential with respect to a circle around the center of rotation), for instance, to the force of friction. Thereby the angular momentum of the test body would be diminished while that of the gravitating body is increased. Another example is presented by the Earth/moon system: The Earth has its own spin about its North-South axis; as a consequence, and due to a hysteresis effect, the tidal bulge (of planet Earth) which is permanently created by the moon's gravity is, when viewed in the plane of rotation of the

Earth/moon system, not strictly symmetrical with respect to the line that connects the center of rotation (of the Earth/moon system) with the center of the moon. This is why Earth's tidal bulge generates a tangential component of the gravitational "force" that acts on the moon. This component increases the moon's angular momentum. Conversely, the gravitational force of the moon acts on the bulge, and thus diminishes the Earth's angular momentum to the same extent to which the angular momentum of the moon is being increased.

By presuming that the angular momentum of the test object stays constant within the Kerr metric, one is thus excluding any tangential force on it a priori. Quod erat demonstrandum.

bb) As opposed to this presupposition, it shall be shown that the gravitational field in the vicinity of a fast spinning spherical object that obeys the Kerr-metric has a tangential component even outside of any horizon.

The formula for a \mathbf{r} -geodesic shall be used as a starting point [see for instance I. Ciufolini, J.A. Wheeler (Ciufolini, Wheeler 1995)(Section 2.4, Eq 2.4.13, p. 30)]:

(1)

$$\frac{d^2 R}{d\tau^2} + \Gamma_{\mu\nu}^1 \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{d^2 R}{d\tau^2} + \Gamma_{\mu\nu}^r \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

The parameter \mathbf{tau} is the proper time of the moving object or of a co-moving observer. The variables \mathbf{x} appearing in Equation (1) shall be defined as follows (using polar coordinates): $\mathbf{x}^0=\mathbf{t}$, $\mathbf{x}^1=\mathbf{r}$, $\mathbf{x}^2=\mathbf{theta}$ (altitude angle measured from the North Pole), $\mathbf{x}^3=\mathbf{phi}$ (azimuth angle). They apply to the reference frame of a stationary observer who sits outside of the gravity field. While \mathbf{r} denotes circumference of a circle around the center of a spherical mass divided by $2\mathbf{pi}$, the variable \mathbf{R} (which has the same orientation in space as \mathbf{r} has) denotes a radial distance measured by laying meter sticks end-to-end. Hence, $d\mathbf{R}$ denotes a short difference in radial distance measured by a local observer who is at rest in the gravity field. It is the second derivative of that \mathbf{R} (and not of \mathbf{r}) with respect to the proper time \mathbf{tau} of a local observer which describes the local acceleration measured by this observer with respect to radially oriented, stationary meter sticks laid end to end (we will return to the distinction between \mathbf{r} and \mathbf{R} later).

cc) Equation (1) says two things:

1) The Christoffel symbol appearing in Equation (1) is an expression of the change in the \mathbf{r} -component of the velocity of freely falling test bodies from one point in spacetime to another along \mathbf{r} or along another direction, to the extent to which it is caused by the properties of spacetime. As will be shown, the velocity of a freely falling test body can be replaced by the velocity of flowing space. Rather than expressing this change (in magnitude of the velocity component \mathbf{v}'_R) by $d\mathbf{v}'_R/d\mathbf{s}'$, it is expressed by $d^2\mathbf{R}/d\mathbf{tau}^2=d\mathbf{v}'_R/d\mathbf{tau}$. Proper velocity \mathbf{v}' , proper spatial distance $d\mathbf{s}'$ and proper time $d\mathbf{tau}$ are the

parameters measured by a local observer in the gravity field who is covering the distance between the two points in spacetime. In case all differential quotients appearing in Equation (1) except $d\mathbf{t}^2/d\mathbf{t}\mathbf{a}\mathbf{u}^2$ are zero, the Christoffel symbol is an expression of the change in the \mathbf{r} -component of the velocity of freely falling test bodies from one point in spacetime to another *along the temporal axis*, that is, as a result of a motion in time alone.

2) In the special case in which an object moving in the \mathbf{r} -direction (with the two other spatial coordinates staying unchanged) is strictly following the direction of the local gravitational field so that the total acceleration it is experiencing along the direction of \mathbf{r} does not differ in magnitude from what is expressed by the Christoffel symbol, its trajectory shall be spoken of as a “ \mathbf{r} -geodesic”. Then the acceleration is exclusively the result of “funny coordinates”, which, in turn, are the result of a local properties of “funny” spacetime.

dd) One has to emphasize the following important fact: When applying Equation (1) in a situation assumed to fall under the category of number 2, one *presupposes* that a test particle in radial motion is on a geodesic, and provides no *proof* for it. The only way to find out whether or not radial motion can be a journey on a geodesic, i.e., can be performed without the intervention of an external force, is the following: One has to start with an assumption in the positive, and then check for conflicts with the laws of physics or for inner contradictions.

b) The situation for non-spinning spherical objects (Schwarzschild metric)

aa) For a better understanding, we will first consider a geodesic in the vicinity of a spherical object that is NOT spinning. For reason of symmetry, we are, in this special case, confident a priori that radial motion can be a journey along a geodesic.

Time \mathbf{t} is the time of a distant observer at rest, whereas $\mathbf{t}\mathbf{a}\mathbf{u}$ is the time of an observer coasting along the supposed geodesic in the gravity field. The angle $\mathbf{t}\mathbf{h}\mathbf{e}\mathbf{t}\mathbf{a}$ is the angle from North Pole to the equator; $\mathbf{p}\mathbf{h}\mathbf{i}$ is the azimuthal angle. The distance \mathbf{r} is circumference divided by $2\mathbf{p}\mathbf{i}$.

The formula for computing the Christoffel symbol [that appears in Equation (1)] is:

(2)

$$\Gamma_{\mu\nu}^p = \frac{g^{pn}}{2} \left(\frac{\delta g_{n\mu}}{\delta x^\nu} + \frac{\delta g_{n\nu}}{\delta x^\mu} - \frac{\delta g_{\mu\nu}}{\delta x^n} \right)$$

Since the motion of the body is confined to the equatorial plane, $\mathbf{t}\mathbf{h}\mathbf{e}\mathbf{t}\mathbf{a}$ is 90° and is fixed. The metric tensor \mathbf{g} (in polar coordinates) is

(3)

$$g_{\mu\nu} = \begin{pmatrix} 1 - \frac{r_s}{r} & 0 & 0 & 0 \\ 0 & -[c^2(1 - \frac{r_s}{r})]^{-1} & 0 & 0 \\ 0 & 0 & -\frac{r^2}{c^2} & 0 \\ 0 & 0 & 0 & -\frac{r^2}{c^2} \sin^2 \theta \end{pmatrix}$$

and is derived from the Schwarzschild metric. More precisely: Equation (3) is derived from:

(4)

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = (1 - \frac{2GM}{c^2 r}) dt^2 - \frac{1}{c^2(1 - \frac{2GM}{c^2 r})} dr^2 - \frac{r^2}{c^2} d\theta^2 - \frac{r^2}{c^2} \sin^2 \theta d\phi^2$$

The right-hand side of Equation (4) is identical with the right-hand side of the Schwarzschild solution. The middle part of Equation (4) is the general expression of a line-element of spacetime. All products \mathbf{gdxdx} have to be summed over \mathbf{mu} und \mathbf{nu} , which are both running from 0 to 1. \mathbf{GM} can be replaced by $\mathbf{r_s c^2 / 2}$, with $\mathbf{r_s}$ denoting the Schwarzschild radius, that is the special distance from the center of the spherical mass at which the escape velocity is \mathbf{c} (speed of light) both in Newtonian physics and in General Relativity.

The equation of the \mathbf{r} -geodesic gives (delta \mathbf{R} is radial distance between two points along a radial line measured by the number of stationary meter sticks laid end to end, whereas \mathbf{r} is circumference divided by $\mathbf{2 \pi i}$; the distinction between these two parameters was stressed by Schwarzschild himself):

(5)

$$\frac{d^2 R}{d\tau^2} + \Gamma_{00}^1 \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} + \Gamma_{33}^1 \frac{dx^3}{d\tau} \frac{dx^3}{d\tau} + \Gamma_{11}^1 \frac{dx^1}{d\tau} \frac{dx^1}{d\tau} = 0$$

or:

(6)

$$\frac{d^2 R}{d\tau^2} + (1 - \frac{r_s}{r}) \frac{c^2 r_s}{2r^2} \frac{dt^2}{d\tau^2} - (1 - \frac{r_s}{r}) r \frac{d\phi^2}{d\tau^2} - [\frac{r_s}{2(1 - r_s/r) r^2}] \frac{dr^2}{d\tau^2} = 0$$

or:

(7)

$$\frac{d^2 R}{d\tau^2} + (1 - \frac{r_s}{r}) \frac{c^2 r_s}{2r^2} \frac{dt^2}{d\tau^2} = (1 - \frac{r_s}{r}) r \frac{d\phi^2}{d\tau^2} + [\frac{r_s}{2(1 - r_s/r) r^2}] \frac{dr^2}{d\tau^2}$$

All other summands of Christoffel-symbols vanish. This is because of:

(8)

$$\Gamma_{00}^1 \frac{dt^2}{d\tau^2} = [\frac{g^{11}}{2} (\frac{\delta g_{10}}{\delta x^0} + \frac{\delta g_{10}}{\delta x^0} - \frac{\delta g_{00}}{\delta x^1})] \frac{dt^2}{d\tau^2}$$

$$= -\frac{c^2}{2} \left(1 - \frac{r_s}{r}\right) \frac{-d(1-r_s/r)}{dr} \frac{dt^2}{d\tau^2} = \left(1 - \frac{r_s}{r}\right) \frac{c^2 r_s}{2r^2} \frac{dt^2}{d\tau^2}$$

$$\Gamma_{11}^1 \frac{dr^2}{d\tau^2} = \left[\frac{g^{11}}{2} \left(\frac{\delta g_{11}}{\delta x^1} + \frac{\delta g_{11}}{\delta x^1} - \frac{\delta g_{11}}{\delta x^1} \right) \right] \frac{dr^2}{d\tau^2} = -\frac{c^2}{2} \left(1 - \frac{r_s}{r}\right) \frac{-d\left[\frac{1}{c^2(1-r_s/r)}\right]}{dr} \frac{dr^2}{d\tau^2}$$

$$= -\frac{c^2}{2} \left(1 - \frac{r_s}{r}\right) \frac{-r_s}{-c^2(1-r_s/r)^2 r^2} \frac{dr^2}{d\tau^2} = -\left[\frac{r_s}{2(1-r_s/r)r^2} \right] \frac{dr^2}{d\tau^2}$$

$$\Gamma_{22}^1 \frac{d\theta^2}{d\tau^2} = 0$$

$$\Gamma_{33}^1 \frac{d\phi^2}{d\tau^2} = \left[\frac{g^{11}}{2} \left(\frac{\delta g_{13}}{\delta x^3} + \frac{\delta g_{13}}{\delta x^3} - \frac{\delta g_{33}}{\delta x^1} \right) \right] \frac{d\phi^2}{d\tau^2} = -\left(1 - \frac{r_s}{r}\right) r \frac{d\phi^2}{d\tau^2}$$

$$\Gamma_{03}^1 \frac{dt}{d\tau} \frac{d\phi}{d\tau} = \left[\frac{g^{11}}{2} \left(\frac{\delta g_{10}}{\delta x^3} + \frac{\delta g_{13}}{\delta x^0} - \frac{\delta g_{03}}{\delta x^1} \right) \right] \frac{dt}{d\tau} \frac{d\phi}{d\tau} = 0$$

$$\Gamma_{30}^1 \frac{d\phi}{d\tau} \frac{dt}{d\tau} = \left[\frac{g^{11}}{2} \left(\frac{\delta g_{13}}{\delta x^0} + \frac{\delta g_{10}}{\delta x^3} - \frac{\delta g_{30}}{\delta x^1} \right) \right] \frac{d\phi}{d\tau} \frac{dt}{d\tau} = 0$$

$$\Gamma_{01}^1 \frac{dt}{d\tau} \frac{dr}{d\tau} = \left[\frac{g^{11}}{2} \left(\frac{\delta g_{10}}{\delta x^1} + \frac{\delta g_{11}}{\delta x^0} - \frac{\delta g_{01}}{\delta x^1} \right) \right] \frac{dt}{d\tau} \frac{dr}{d\tau} = 0$$

$$\Gamma_{10}^1 \frac{dr}{d\tau} \frac{dt}{d\tau} = \left[\frac{g^{11}}{2} \left(\frac{\delta g_{11}}{\delta x^0} + \frac{\delta g_{10}}{\delta x^1} - \frac{\delta g_{10}}{\delta x^1} \right) \right] \frac{dr}{d\tau} \frac{dt}{d\tau} = 0$$

$$\Gamma_{20}^1 \frac{d\theta}{d\tau} \frac{dt}{d\tau} = 0$$

$$\Gamma_{02}^1 \frac{dt}{d\tau} \frac{d\theta}{d\tau} = 0$$

$$\Gamma_{21}^1 \frac{d\theta}{d\tau} \frac{dr}{d\tau} = 0$$

$$\Gamma_{12}^1 \frac{dr}{d\tau} \frac{d\theta}{d\tau} = 0$$

$$\Gamma_{13}^1 \frac{dr}{d\tau} \frac{d\phi}{d\tau} = \left[\frac{g^{11}}{2} \left(\frac{\delta g_{11}}{\delta x^3} + \frac{\delta g_{13}}{\delta x^1} - \frac{\delta g_{13}}{\delta x^1} \right) \right] \frac{dr}{d\tau} \frac{d\phi}{d\tau} = 0$$

$$\Gamma_{31}^1 \frac{d\phi}{d\tau} \frac{dr}{d\tau} = \left[\frac{g^{11}}{2} \left(\frac{\delta g_{13}}{\delta x^1} + \frac{\delta g_{11}}{\delta x^3} - \frac{\delta g_{31}}{\delta x^1} \right) \right] \frac{d\phi}{d\tau} \frac{dr}{d\tau} = 0$$

$$\Gamma_{23}^1 \frac{d\theta}{d\tau} \frac{d\phi}{d\tau} = 0$$

$$\Gamma_{32}^3 \frac{d\phi}{d\tau} \frac{d\theta}{d\tau} = 0$$

When multiplying both sides of Equation (7) by $d\mathbf{tau}^2/dt^2$, $1/r$ and by $(\mathbf{1}-\mathbf{r}_s/r)^{-1}$, Equation (7) converts into:

(9)

$$\frac{d^2 R}{d\tau^2} \frac{d\tau^2}{dt^2} \frac{1}{r(1-r_s/r)} = -\frac{c^2 r_s}{2r^3} + \frac{r_s}{2(1-r_s/r)^2 r^3} \frac{dr^2}{dt^2} + \frac{d\phi^2}{dt^2}$$

If we set $d\mathbf{phi}/dt=0$, Equation (9) converts into:

(10)

$$\frac{d^2 R}{d\tau^2} \frac{d\tau^2}{dt^2} \frac{1}{r(1-r_s/r)} = -\frac{c^2 r_s}{2r^3} + \frac{r_s}{2(1-r_s/r)^2 r^3} \frac{dr^2}{dt^2}$$

For the raising of indices of tensor components [of the 2 x 2 metric tensor \mathbf{g} , as required by Equation (3)], the following equations have been observed:

(11)

$$g^{\mu\nu} g_{\nu\mu} = \delta_n^\mu \quad , \quad g^{\mu\nu} g_{\mu\nu} = 1 + 1 = 2$$

The **delta** is the Kronecker-delta, which in our case is a 2 x 2 tensor whose two diagonal components are both unity, and whose two off-diagonal components are both zero. The second equation is an expression of the fact that, as a result of the Schwarzschild metric, all off-diagonal components of the metric tensor \mathbf{g} (with low indices) are zero. This makes it easy to determine what tensor is yielded by the raising of indices: One simply has to form the inverse of each component.

Since, according to the Schwarzschild metric, $d\mathbf{tau}^2/dt^2$ times $(\mathbf{1}-\mathbf{r}_s/r)^{-1}$ equals unity (given \mathbf{tau} is the time of an at least momentarily stationary observer in the gravity field), and since we now imagine $d\mathbf{r}/dt$ to be zero (for a moment) when applied to a (at least momentarily) stationary object located next to our stationary observer in the gravity field, we get for this situation from Equation (10):

(12)

$$\frac{d^2 R}{d\tau^2} = -\frac{c^2 r_s}{2r^2} = -\frac{MG}{r^2} = g'$$

or:

(13)

$$\frac{d^2 R}{dt^2} = -\frac{c^2 r_s}{2r^2} (1 - \frac{r_s}{r}) = -\frac{MG}{r^2} (1 - \frac{r_s}{r})$$

\mathbf{M} is the mass of the spherical body, \mathbf{G} is Newton's gravitational constant, \mathbf{g}' is the local gravitational acceleration measured by an observer who is at rest in the gravity field. Equation (12) can also be written as follows:

(14)

$$\frac{d^2 R}{d\tau^2} = -\Gamma_{00}^1 \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} = -\left(1 - \frac{r_s}{r}\right) \frac{c^2 r_s}{2r^2} \frac{dt^2}{d\tau^2} = -\frac{c^2 r_s}{2r^2}$$

The right-hand side of Equation (14) presents itself as the 1-00- summand of the Christoffel symbol, multiplied by the appropriate differential quotient $d\mathbf{t}^2/d\mathbf{t}\mathbf{a}\mathbf{u}^2$. Here the Christoffel symbol is an expression of the change in the \mathbf{r} -component of the velocity of freely falling test bodies from one point in space-time to another *along the temporal axis*, that is, as a result of a motion in time alone. (How such a change can occur despite the fact that the gravitational field is static shall be explained in the appendix).

Equation (12) (valid for the observer in the gravity field) is identical with Newton's equation for the gravitational force. Since we arrived at the (Newtonian) law of the gravitational "force" on the sole basis of the assumption contained in (1) that no real force is active and that any acceleration is the result of "funny coordinates" (which, in turn, is the result of "funny" space-time), we have obtained a strict proof of gravity *not* being a force in General Relativity.

bb) In order to make sure that the choice of $d^2\mathbf{R}$ (rather than $d^2\mathbf{r}$) in (1) and hence in (12) and (13) is justified (\mathbf{R} is radial distance measured by meter sticks laid end to end, \mathbf{r} is circumference of a circle, divided by 2π), let us re-consider (10):

(15)

$$\frac{d^2 R}{d\tau^2} \frac{d\tau^2}{dt^2} \frac{1}{r(1 - r_s/r)} = -\frac{c^2 r_s}{2r^3} + \frac{r_s}{2(1 - r_s/r)^2 r^3} \frac{dr^2}{dt^2}$$

This equation can be re-arranged in order to present itself as follows, given that $\mathbf{t}\mathbf{a}\mathbf{u}$ is the time of an observer who is (momentarily) at rest in the gravity field, $d\mathbf{r}/d\mathbf{t}$ (which is no longer presupposed to be zero) is the velocity of an object in free fall in the reference frame of a distant observer Bob outside the gravity field, $\mathbf{v}' = d\mathbf{R}/d\mathbf{t}\mathbf{a}\mathbf{u}$ is the velocity of that object in the reference frame of the observer Alice (whose time is $\mathbf{t}\mathbf{a}\mathbf{u}$) who is momentarily at rest in the gravity field, and who is watching the object passing by:

(16)

$$\frac{d^2 R}{d\tau^2} = \left[-\frac{c^2 r_s}{2r^2} + \frac{r_s}{2(1 - r_s/r)^2 r^2} \frac{dr^2}{dt^2} \right] \left(1 - \frac{r_s}{r}\right) \frac{dt^2}{d\tau^2} = -\frac{c^2 r_s}{2r^2} + \frac{(v')^2 r_s}{2r^2}$$

Since we presupposed that no real force is acting on objects or observers inside the gravity field, the momentary rest of the observer Alice has to be the momentary rest at a turning point of an upward motion (apex), caused by a previously performed upward throw the observer Alice was subject to. The term $(1 - r_s/r)^{-2} d\mathbf{r}^2/d\mathbf{t}^2$ is equal to $(\mathbf{v}')^2$ (radial velocities measured by stationary Alice are affected both by the contraction of meter sticks and by the dilation of time, see below). The term $(1 - r_s/r) d\mathbf{t}^2/d\mathbf{t}\mathbf{a}\mathbf{u}^2$ is, according to the Schwarzschild equation, equal to unity for events (i.e., the ticks of a

clock) on the world line of any stationary observer in the gravity field

According to Equation (16), an object whose radial velocity is \mathbf{c} does not experience any more radial acceleration. Consequently, radial velocities that are below \mathbf{c} in the beginning cannot exceed the local speed of light outside of the Schwarzschild radius.

Moreover, Equation (16) shows that even far away from the gravitating mass, that is at a large distance \mathbf{r} from the mass, an object cannot be in possession of a velocity greater than \mathbf{c} to start with: If the velocity \mathbf{v}' of an object at a large distance \mathbf{r} were greater than \mathbf{c} , the acceleration $\mathbf{d}^2\mathbf{R}/\mathbf{d}\tau^2$ would be positive, meaning that the object would be decelerating because of a reversal of the gravitational “force”. But this could not be a physically valid statement. Instead, one has to conclude that velocities greater than \mathbf{c} are physically impossible even in almost flat spacetime, that is, in Special Relativity.

The fact that, according to Equation (16), $\mathbf{d}^2\mathbf{R}/\mathbf{d}\tau^2$ is (as expected) always negative or zero has a further consequence: Imagine we would have written $\mathbf{d}^2\mathbf{r}/\mathbf{d}\tau^2$ instead of $\mathbf{d}^2\mathbf{R}/\mathbf{d}\tau^2$ in Equation (1) and hence in Equation (16). Then Equation (16) would turn into (with $\mathbf{d}\tau/\mathbf{d}t = \sqrt{1-\mathbf{v}^2/\mathbf{c}^2}$):

$$\frac{d^2r}{dt^2} = \left(1 - \frac{r_s}{r}\right) \frac{d^2r}{dt^2} = -\frac{c^2 r_s}{2r^2} + \frac{(v')^2 r_s}{2r^2} \Rightarrow \frac{d^2r}{dt^2} = \frac{1}{1 - \frac{r_s}{r}} \left[-\frac{c^2 r_s}{2r^2} + \frac{(v')^2 r_s}{2r^2}\right]$$

As before, the right-hand side of the equation would have to be always negative, given $(\mathbf{v}')^2 < \mathbf{c}^2$. But a freely falling object, when watched from outside of the gravity field, is gathering speed only to start with, and is slowing down near the Schwarzschild radius. As will be shown below, the velocity of a freely falling test object in the reference frame of a distant observer (Bob) who sits outside of the gravity field, that is, $\mathbf{d}^2\mathbf{r}/\mathbf{d}t^2$, is increasing until it reaches a position $3\mathbf{r}_s$ away from the center of mass (where its speed is $0.3849\mathbf{c}$), and then slows down in front of the Schwarzschild horizon. This is why $\mathbf{d}^2\mathbf{r}/\mathbf{d}t^2$ is not (different from $\mathbf{d}^2\mathbf{R}/\mathbf{d}t^2$) always negative [given $(\mathbf{1}-\mathbf{r}_s/\mathbf{r})$ is always positive]. Instead, $\mathbf{d}^2\mathbf{r}/\mathbf{d}t^2$ changes sign from negative to positive somewhere between the starting point of the free fall and the Schwarzschild radius. But this is incompatible with right-hand side of (16) or (17). In order to avoid this inconsistency, $\mathbf{d}^2\mathbf{R}$, and not $\mathbf{d}^2\mathbf{r}$, had to be used in Equation (1) and in Equation (16) on the left-hand sides. Quod erat demonstrandum.

c) The situation for spinning spherical objects in the equatorial plane (Kerr metric)

aa) Let us now follow the same train of thought for a *spinning* spherical body. For a moment, we shall be assuming that a motion along an anti-radial trajectory can be a motion along a geodesic.

The metric tensor for motions confined to two dimensions (time as the first, and a radial direction \mathbf{r} in the equatorial plane as the second dimension) would then be (Kerr metric in Boyer-Lindquist coordinates, with θ being 90°):

$$g_{\mu\nu} = \begin{pmatrix} 1 - \frac{r_s}{r} & 0 \\ 0 & -[c^2(1 - \frac{r_s}{r} + \frac{a^2}{r^2})]^{-1} \end{pmatrix}$$

Equation (18) is derived from:

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = (1 - \frac{r_s}{r})dt^2 - \frac{1}{c^2(1 - \frac{r_s}{r} + \frac{a^2}{r^2})}dr^2$$

where the products \mathbf{gdxdx} have to be summed over μ and ν , which are both running from 0 to 1. The resulting sum in Equation (19) (which has two non-zero summands only) is identical with the (shortened) right-hand side of the equation of the Kerr metric for a spinning Black Hole [see L. Rezolla, O. Zanotti (Rezolla, Zanotti 2013) (Chapter 1.7.2, Eq. 1.251, p. 55)]. The term a is defined as:

$$a = \frac{J}{Mc}$$

with \mathbf{J}/\mathbf{M} being angular momentum per unit mass of the spherical body [see Rezolla, Zanotti 2013 (p. 55): "... leads to the straightforward interpretation of the parameter a as the angular momentum per unit mass, $a := J/M$ "].

For the assumed \mathbf{r} -geodesic, we now have as a first summand in (1):

$$\begin{aligned} \Gamma_{00}^1 \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} &= \left[\frac{g^{10}}{2}(\dots) + \frac{g^{11}}{2} \left(\frac{\delta g_{10}}{\dots} + \frac{\delta g_{10}}{\dots} - \frac{\delta g_{00}}{\delta x_1} \right) \right] \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} \\ &= \frac{c^2}{2} \left(1 - \frac{r_s}{r} + \frac{a^2}{r^2} \right) \frac{d(1 - r_s/r)}{dr} \frac{dt^2}{d\tau^2} = \left(1 - \frac{r_s}{r} + \frac{a^2}{r^2} \right) \frac{c^2 r_s}{2r^2} \frac{dt^2}{d\tau} \end{aligned}$$

As a second summand we have:

$$\begin{aligned} \Gamma_{11}^1 \frac{dx^1}{d\tau} \frac{dx^1}{d\tau} &= \left[\frac{g^{10}}{2}(\dots) + \frac{g^{11}}{2} \left(\frac{dg^{11}}{dx^1} + \frac{dg^{11}}{dx^1} - \frac{dg^{11}}{dx^1} \right) \right] \frac{dx^1}{d\tau} \frac{dx^1}{d\tau} \\ &= - \left[\frac{c^2}{2} \left(1 - \frac{r_s}{r} + \frac{a^2}{r^2} \right) \frac{d\left(-\frac{1}{1 - r_s/r + a^2/r^2} \right)}{c^2 dr} \right] \frac{dr^2}{d\tau^2} = - \left[\frac{c^2}{2} \left(1 - \frac{r_s}{r} + \frac{a^2}{r^2} \right) \frac{\frac{r_s}{r^2} - \frac{2a^2}{r^3}}{c^2 \left(-\frac{r_s}{r} + \frac{a^2}{r^2} + 1 \right)^2} \right] \frac{dr^2}{d\tau^2} \end{aligned}$$

We then get from Equations (1),(20) and (21):

(22)

$$\frac{d^2 R}{d\tau^2} + \left(1 - \frac{r_s}{r} + \frac{a^2}{r^2}\right) \frac{c^2 r_s}{2r^2} \frac{dt^2}{d\tau^2} - \frac{c^2}{2} \left(1 - \frac{r_s}{r} + \frac{a^2}{r^2}\right) \frac{\frac{r_s}{r^2} - \frac{2a^2}{r^3}}{c^2 \left(-\frac{r_s}{r} + \frac{a^2}{r^2} + 1\right)^2} \frac{dr^2}{d\tau^2} = 0$$

After multiplication by $d\mathbf{tau}^2/dt^2$, $\mathbf{1}/r$, and $\mathbf{1}/(\mathbf{1}-r_s/r)$, Equation (22) turns into:

(23)

$$\frac{d^2 R}{d\tau^2} \frac{d\tau^2}{dt^2} \frac{1}{r(1 - \frac{r_s}{r})} = - \frac{(1 - \frac{r_s}{r} + \frac{a^2}{r^2}) c^2 r_s}{(1 - \frac{r_s}{r})^2 r^3} + \frac{c^2 (1 - \frac{r_s}{r} + \frac{a^2}{r^2})}{2r (1 - \frac{r_s}{r})} \frac{\frac{r_s}{r^2} - \frac{2a^2}{r^3}}{c^2 \left(-\frac{r_s}{r} + \frac{a^2}{r^2} + 1\right)^2} \frac{dr^2}{dt^2}$$

In order to simplify the equation, we introduce a variable \mathbf{h} :

(24)

$$h = \frac{1 - \frac{r_s}{r} + \frac{a^2}{r^2}}{1 - \frac{r_s}{r}} = 1 + \frac{a^2}{r^2(1 - \frac{r_s}{r})}$$

and hence get

(25)

$$\frac{d^2 R}{d\tau^2} \frac{d\tau^2}{dt^2} \frac{1}{r(1 - \frac{r_s}{r})} = - h \frac{c^2 r_s}{2r^3} + \frac{h}{2r} \frac{\frac{r_s}{r^2} - \frac{2a^2}{r^3}}{\left(-\frac{r_s}{r} + \frac{a^2}{r^2} + 1\right)^2} \frac{dr^2}{dt^2}$$

For further simplification, we introduce another variable which we call \mathbf{j} :

(26)

$$\frac{1}{2r} \frac{\frac{r_s}{r^2} - \frac{2a^2}{r^3}}{\left(-\frac{r_s}{r} + \frac{a^2}{r^2} + 1\right)^2} = j \frac{r_s}{2(1 - \frac{r_s}{r})^2 r^3} \Rightarrow j = \frac{(1 - \frac{r_s}{r})^2 r^2}{r} \frac{\frac{r_s}{r^2} - \frac{2a^2}{r^3}}{\left(-\frac{r_s}{r} + \frac{a^2}{r^2} + 1\right)^2}$$

Our equation (25) then turns into:

(27)

$$\frac{d^2 R}{d\tau^2} \frac{d\tau^2}{dt^2} \frac{1}{r(1 - \frac{r_s}{r})} = - h \frac{c^2 r_s}{2r^3} + h j \frac{r_s}{2(1 - \frac{r_s}{r})^2 r^3} \frac{dr^2}{dt^2}$$

This equation can be re-arranged in order to present itself as follows, given that \mathbf{tau} is the time of an observer (Alice) who is momentarily at rest in the gravity field, $d\mathbf{r}/d\mathbf{t}$ is the anti-radial velocity component of an object in free fall in the frame of a distant observer (Bob) outside the gravity field, $\mathbf{v}' =$

$d\mathbf{R}/d\tau$ is the anti-radial velocity component of that object in the frame of the observer Alice (whose time is τ) who is momentarily at rest in the gravity field (at distance \mathbf{r}), and who is watching the object passing by closely: (28)

$$\begin{aligned} \frac{d^2 R}{d\tau^2} &= \left[-h \frac{c^2 r_s}{2r^2} + h j \frac{k r_s}{2k(1-\frac{r_s}{r})^2 r^2} \frac{dr^2}{dt^2} \right] \left(1 - \frac{r_s}{r} \right) \frac{dt^2}{d\tau^2} \\ &= -h \frac{c^2 r_s}{2r^2} + h j \frac{k r_s}{2(1-\frac{r_s}{r} + \frac{a^2}{r^2})(1-\frac{r_s}{r})r^2} \frac{dr^2}{dt^2} = h \left(-\frac{c^2 r_s}{2r^2} + j \frac{(v')^2 k r_s}{2r^2} \right) \end{aligned}$$

In Equation (28) we have introduced a third variable, \mathbf{k} , for which we suitably define:

$$(29) \quad k \left(1 - \frac{r_s}{r} \right) = 1 - \frac{r_s}{r} + \frac{a^2}{r^2} \Rightarrow k = \frac{1 - \frac{r_s}{r} + \frac{a^2}{r^2}}{1 - \frac{r_s}{r}}$$

The new variable \mathbf{k} is identical to the variable \mathbf{h} .

The term $(\mathbf{1}-\mathbf{r}_s/\mathbf{r}+\mathbf{a}^2/\mathbf{r}^2)^{-1}(\mathbf{1}-\mathbf{r}_s/\mathbf{r})^{-1}d\mathbf{r}^2/dt^2$ appearing in the middle part of Equation (28) is equal to $(\mathbf{v}')^2$, since radial velocities are affected both by the contraction of meter sticks and by the dilation of time (see below). In the Kerr metric for the equatorial plane, the spatial contraction factor (in the radial direction) is different from the temporal contraction factor. This has been given due consideration. Moreover, the term $(\mathbf{1}-\mathbf{r}_s/\mathbf{r}) dt^2/d\tau^2$ in Equation (28) is equal to unity according to the Kerr metric in the equatorial plane (for events on the world line of any stationary observer in the gravity field, e.g., for ticks of Alice's watch).

bb) We have now reached a crucial point: When we arrived at Equation (28), we had assumed that a body in free anti-radial fall is on a geodesic. That is to say: We assumed that its total acceleration in the anti-radial direction is identical in magnitude with the acceleration expected as a result of “funny coordinates” and hence identical with the magnitude of the Christoffel symbol in Equation (1). But we did not know whether or not the assumption was justified. The only way to get certainty is the following (as mentioned above): We have to check the consequences of our assumption for inconsistencies. If it turns out that there are no conflicts with other laws of physics, especially those of relativity, we will be confident that a motion along \mathbf{r} is capable of being a motion along a geodesic. If, on the contrary, we will find that we run into conflicts with other laws of physics, we will be sure that the radial motion CANNOT be motion along a geodesic, but needs the application of an external force in order to perform it.

When we dealt with the Schwarzschild metric (for a non-spinning spherical body), Equation (16) was a check of that kind. Given that no object can surpass the speed of light (at least not while it is outside of a black hole), Equation (16) had to guarantee the keeping of that speed limit. If it hadn't, we would have been sure that a radial motion cannot be a motion along a geodesic. We know that the outcome of Equation (16) was in the positive, and we obtained a confirmation of our assumption that a radial motion can be a motion along a geodesic (when dealing with non-spinning spherical bodies).

But when applying Equation (28), the test fails. In case $\mathbf{d}^2\mathbf{R}/\mathbf{d}\tau^2$ is set to zero, Equation (28) turns into: $\mathbf{v}' = (\mathbf{jk})^{-1/2}\mathbf{c}$. If no local speed is to surpass the speed of light \mathbf{c} , it would require that $\mathbf{jk}=\mathbf{1}$. However, from Equations (26) and (29) we get:

$$(30) \quad \begin{aligned} jk &= \frac{(1-\frac{r_s}{r})^2 r^2}{r_s} \frac{\frac{r_s}{r^2} - \frac{2a^2}{r^3}}{(-\frac{r_s}{r} + \frac{a^2}{r^2} + 1)^2} \frac{1 - \frac{r_s}{r} + \frac{a^2}{r^2}}{1 - \frac{r_s}{r}} \\ &= \frac{r^2 (1-\frac{r_s}{r}) (\frac{r_s}{r^2} - \frac{2a^2}{r^3})}{r_s (1 - \frac{r_s}{r} + \frac{a^2}{r^2})} < 1 \quad \text{if } a^2 > 0 \end{aligned}$$

The situation can also be described as follows. Equation (28) turns into:

$$(31) \quad \frac{d^2 R}{d\tau^2} < 0 \quad \text{for } v' = c \quad \wedge \quad a^2 > 0$$

instead of

$$(32) \quad \frac{d^2 R}{d\tau^2} = 0 \quad \text{for } v' = c \quad \wedge \quad a^2 > 0$$

which would be expected in case a radial trajectory could be a geodesic.

How can the result presented in Equation (31) be accounted for? The only explanation is: One has to keep in mind that Equation (31) may not yield the *resulting* acceleration, given it only yields that fraction of the resulting acceleration which is due to “funny spacetime”. Hence, in the situation described by Equation (31), one can still be sure that the *resulting* acceleration $\mathbf{d}^2\mathbf{R}/\mathbf{d}\tau^2$ (of an object whose local velocity \mathbf{v}' is \mathbf{c} or very close to \mathbf{c}) is zero after all, and it's only the acceleration brought about by “funny spacetime” which is different from zero. But this can only be due to the fact that the negative acceleration in the \mathbf{r} -direction yielded by Equation (31) as an effect of “funny” spacetime is partly offset by some force acting in the counter-direction.

This counter-force can only be the force of inertia. Whenever the resulting acceleration of a test body has a component transverse to the gravitational field

lines, it has to cope with an relativistic increase in mass with speed, whereas the acceleration brought about by “funny” spacetime (that is, the acceleration experienced when traveling along a geodesic) has not.

A similar situation arises when an electron traveling at almost the speed of light is pushed forward by an electric field: The accelerating electric force does not lead to a further increase in speed of the electron, but only to an increase in its mass, whose magnitude exceeds all limits when the velocity approaches c . The electric force is thus offset by the force of inertia, whose direction is opposed to that of the electric force.

The same can be true when an object subject to a downward gravitational “force” gathers speed along a ramp and is thus forced to acquire a velocity component transverse to the gravitational field. Obviously, this is the situation faced by (28) and (31).

From this we conclude: The radial motion of an object CANNOT be a motion along a geodesic. Then, however, it follows that the gravitational field lines cannot be strictly radial. Given they are not strictly radial, they must have a tangential component. *Quod erat demonstrandum*.

cc) The recognition of a curved geodesic was foreshadowed in 1979 in an article by N.A. Sharp (Sharp 1979)[Section 6.5 – Radial (constant theta) geodesics –, p. 667]. Sharp argued that “*because of frame dragging*”, geodesics in the equatorial plane could not be strictly radial, but have to be more hyperboloidal:

“Geodesics at constant theta have varying phi because of frame dragging, and are therefore not strictly radial, being more hyperboloidal.”

However, Sharp did not provide any proof or reference. Nor did Sharp mention the non-conservativeness of the gravitational field which follows from this.

2. Gravitational “forces” transverse to the radial direction as revealed by the equation of a phi-geodesic

Although we have already found that an anti-radial trajectory of an object in the gravity field of a spinning spherical mass cannot be a geodesic and must hence be subject to a transverse gravitational “force”, we will scrutinize the transverse force somewhat further.

a) The equation of a phi-geodesic in the Kerr metric

Let us consider the equation of a **phi**-geodesic. The equation of the **phi**-geodesic is:

(32)

$$\frac{d^2 x^3}{d\tau^2} + \Gamma_{\mu\nu}^3 \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \frac{d^2 \phi}{d\tau^2} + \Gamma_{\mu\nu}^\phi \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

The Christoffel symbol reads:

(33)

$$\Gamma_{\mu\nu}^{\phi} = \Gamma_{\mu\nu}^3 = \Gamma_{\mu\nu}^p = \frac{g^{pn}}{2} \left(\frac{\delta g_{n\mu}}{\delta x^{\nu}} + \frac{\delta g_{n\nu}}{\delta x^{\mu}} - \frac{\delta g_{\mu\nu}}{\delta x^n} \right)$$

The variable \mathbf{x} (of which there are four, that is three for spatial coordinates and one for the temporal coordinate) denotes the coordinates of an event in spacetime, and runs from \mathbf{x}^0 to \mathbf{x}^3 . Since polar coordinates are used, we have: $\mathbf{x}^0 = \mathbf{t}$, $\mathbf{x}^1 = \mathbf{r}$, $\mathbf{x}^2 = \mathbf{theta}$, $\mathbf{x}^3 = \mathbf{phi}$.

As is revealed by the equation of the **phi**-geodesic, the Christoffel symbol is an expression of the change in the **phi**-component of the velocity of freely falling test bodies (all of which come from afar) with proper time **tau** when moving along **phi** or along another direction.

The Kerr metric for spatial displacements in the equatorial plane is:

(34)

$$\begin{aligned} d\tau^2 &= g_{\mu\nu} dx^{\mu} dx^{\nu} \\ &= \left(1 - \frac{r_s}{r}\right) dt^2 - \frac{1}{c^2 \left(1 - \frac{r_s}{r} + \frac{a^2}{r^2}\right)} dr^2 - 0 d\theta^2 - \frac{1}{c^2} \left(r^2 + a^2 + \frac{r_s a^2}{r}\right) d\phi^2 + \frac{2r_s a}{rc} d\phi dt \end{aligned}$$

From this follows for the metric tensor \mathbf{g} in the equatorial plane:

(35)

$$g_{\mu\nu} = \begin{array}{ccccc} 1 - \frac{r_s}{r} & 0 & 0 & \frac{r_s a}{rc} \\ 0 & -\left[c^2 \left(1 - \frac{r_s}{r} + \frac{a^2}{r^2}\right)\right]^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r_s a}{rc} & 0 & 0 & -\frac{1}{c^2} \left(r^2 + a^2 + \frac{r_s a^2}{r}\right) \end{array}$$

The summands that make up the Christoffel symbol thus are:

(36)

$$\Gamma_{00}^3 \frac{dt^2}{d\tau^2} = \left[\frac{g^{30}}{2} \left(\frac{\delta g_{00}}{\delta x^0} + \frac{\delta g_{00}}{\delta x^0} - \frac{\delta g_{00}}{\delta x^0} \right) + \frac{g^{33}}{2} \left(\frac{\delta g_{30}}{\delta x^0} + \frac{\delta g_{30}}{\delta x^0} - \frac{\delta g_{00}}{\delta x^3} \right) \right] \frac{dt^2}{d\tau^2} = 0$$

$$\Gamma_{11}^3 \frac{dr^2}{d\tau^2} = \left[\frac{g^{30}}{2} \left(\frac{\delta g_{01}}{\dots} + \frac{\delta g_{01}}{\dots} - \frac{\delta g_{11}}{\delta x^0} \right) + \frac{g^{33}}{2} \left(\frac{\delta g_{31}}{\dots} + \frac{\delta g_{31}}{\dots} - \frac{\delta g_{11}}{\delta x^3} \right) \right] \frac{dr^2}{d\tau^2} = 0$$

$$\Gamma_{22}^3 \frac{d\theta^2}{d\tau^2} = 0$$

$$\Gamma_{33}^3 \frac{d\phi^2}{d\tau^2} = \left[\frac{g^{30}}{2} \left(\frac{\delta g_{03}}{\delta x^3} + \frac{\delta g_{03}}{\delta x^3} - \frac{\delta g_{33}}{\delta x^0} \right) + \frac{g^{33}}{2} \left(\frac{\delta g_{33}}{\delta x^3} + \frac{\delta g_{33}}{\delta x^3} - \frac{\delta g_{33}}{\delta x^3} \right) \right] \frac{d\phi^2}{d\tau^2} = 0$$

$$\Gamma_{03}^3 \frac{dt}{d\tau} \frac{d\phi}{d\tau} = \left[\frac{g^{30}}{2} \left(\frac{\delta g_{00}}{\delta x^3} + \frac{\delta g_{03}}{\delta x^0} - \frac{\delta g_{03}}{\delta x^0} \right) + \frac{g^{33}}{2} \left(\frac{\delta g_{30}}{\delta x^0} + \frac{\delta g_{33}}{\delta x^0} - \frac{\delta g_{03}}{\delta x^3} \right) \right] \frac{dt}{d\tau} \frac{d\phi}{d\tau} = 0$$

$$\Gamma_{30}^3 \frac{d\phi}{d\tau} \frac{dt}{d\tau} = \left[\frac{g^{30}}{2} \left(\frac{\delta g_{03}}{\delta x^0} + \frac{\delta g_{00}}{\delta x^3} - \frac{\delta g_{30}}{\delta x^0} \right) + \frac{g^{33}}{2} \left(\frac{\delta g_{33}}{\delta x^0} + \frac{\delta g_{30}}{\delta x^3} - \frac{\delta g_{30}}{\delta x^3} \right) \right] \frac{d\phi}{d\tau} \frac{dt}{d\tau} = 0$$

$$\Gamma_{01}^3 \frac{dt}{d\tau} \frac{dr}{d\tau} = \left[\frac{g^{30}}{2} \frac{\delta g_{00}}{\delta x^1} + g^{\frac{33}{2}} \frac{\delta g_{30}}{\delta x^1} \right] \frac{dt}{d\tau} \frac{dr}{d\tau} \neq 0$$

$$\Gamma_{10}^3 \frac{dr}{d\tau} \frac{dt}{d\tau} = \left[\frac{g^{30}}{2} \frac{\delta g_{00}}{\delta x^1} + g^{\frac{33}{2}} \frac{\delta g_{30}}{\delta x^1} \right] \frac{dr}{d\tau} \frac{dt}{d\tau} \neq 0$$

$$\Gamma_{20}^3 \frac{d\theta}{d\tau} \frac{dt}{d\tau} = 0$$

$$\Gamma_{02}^3 \frac{dt}{d\tau} \frac{d\theta}{d\tau} = 0$$

$$\Gamma_{21}^3 \frac{d\theta}{d\tau} \frac{dr}{d\tau} = 0$$

$$\Gamma_{12}^3 \frac{dr}{d\tau} \frac{d\theta}{d\tau} = 0$$

$$\Gamma_{13}^3 \frac{dr}{d\tau} \frac{d\phi}{d\tau} = \left[\frac{g^{30}}{2} \frac{\delta g_{03}}{\delta x^1} + \frac{g^{33}}{2} \frac{\delta g_{33}}{\delta x^1} \right] \frac{dr}{d\tau} \frac{d\phi}{d\tau} \neq 0$$

$$\Gamma_{31}^3 \frac{d\phi}{d\tau} \frac{dr}{d\tau} = \left[\frac{g^{30}}{2} \frac{\delta g_{03}}{\delta x^1} + \frac{g^{33}}{2} \frac{\delta g_{33}}{\delta x^1} \right] \frac{d\phi}{d\tau} \frac{dr}{d\tau} \neq 0$$

$$\Gamma_{23}^3 \frac{d\theta}{d\tau} \frac{d\phi}{d\tau} = 0$$

$$\Gamma_{32}^3 \frac{d\phi}{d\tau} \frac{d\theta}{d\tau} = 0$$

One realizes: In the Kerr metric for the equatorial plane, all components of the metric tensor except \mathbf{g}_{00} , \mathbf{g}_{11} , \mathbf{g}_{33} , \mathbf{g}_{30} and \mathbf{g}_{03} are zero. These components depend on \mathbf{r} ($=\mathbf{x}^1$), and on nothing else. Also zero are all differential quotients in the Christoffel symbol which contain \mathbf{dtheta} in the numerator (given we restrict ourselves to motions in the equatorial plane).

b) Differences between the phi-geodesics in the Kerr metric and the Schwarzschild metric, and what these differences reveal

aa) Before going deeper into those summands which are non-zero, let us ask which summands are non-zero in case the Schwarzschild metric – and not the Kerr metric – were applicable. The answer is: Only two summands of the

four would be left. These two summands would be:

(37)

$$\Gamma_{13}^3 \frac{dr}{d\tau} \frac{d\phi}{d\tau} = \left[\frac{g^{30}}{2} \frac{\delta g_{03}}{\delta x^1} + \frac{g^{33}}{2} \frac{\delta g_{33}}{\delta x^1} \right] \frac{dr}{d\tau} \frac{d\phi}{d\tau} \neq 0$$

$$\Gamma_{31}^3 \frac{d\phi}{d\tau} \frac{dr}{d\tau} = \left[\frac{g^{30}}{2} \frac{\delta g_{03}}{\delta x^1} + \frac{g^{33}}{2} \frac{\delta g_{33}}{\delta x^1} \right] \frac{d\phi}{d\tau} \frac{dr}{d\tau} \neq 0$$

It is the term $\mathbf{g}^{33} \mathbf{d}g_{33}/2\mathbf{d}x^1$ in Equation (37) which does not vanish here. That is to say: In the Schwarzschild metric of a non-spinning spherical mass, a tangential acceleration $\mathbf{d}^2\mathbf{phi}/\mathbf{d}\tau^2$ (of a test body) different in magnitude from zero would occur only in case a motion of a test object would be performed in the anti-radial and in the tangential direction combined.

This can be easily explained by the Kepler's second law. The angular velocity of an orbiter on an elliptic trajectory is not constant (even though the orbiter is not subject to a tangential force), but is faster near the perihelion than it is near the aphelion. This is why $\mathbf{d}^2\mathbf{phi}/\mathbf{d}\tau^2$ is different from zero on an elliptic orbit.

Of course, in case $\mathbf{d}\mathbf{phi}/\mathbf{d}\tau=0$, all sides of the Equation (37) vanish to zero even when $\mathbf{d}\mathbf{r}/\mathbf{d}\tau$ does not vanish. That is to say: A strictly anti-radial motion does not cause any transverse acceleration in the Schwarzschild metric (as expected).

bb) In the Kerr metric, the situation is different. Even in case of $\mathbf{d}\mathbf{phi}/\mathbf{d}\tau=0$, two summands do not vanish as long as $\mathbf{d}\mathbf{r}/\mathbf{d}\tau$ is different from zero. These summands are:

(38)

$$\Gamma_{01}^3 \frac{dt}{d\tau} \frac{dr}{d\tau} = \left[\frac{g^{30}}{2} \frac{\delta g_{00}}{\delta x^1} + \frac{g^{33}}{2} \frac{\delta g_{30}}{\delta x^1} \right] \frac{dt}{d\tau} \frac{dr}{d\tau} = - \frac{d^2\phi}{d\tau^2} \neq 0$$

$$\Gamma_{10}^3 \frac{dr}{d\tau} \frac{dt}{d\tau} = \left[\frac{g^{30}}{2} \frac{\delta g_{00}}{\delta x^1} + \frac{g^{33}}{2} \frac{\delta g_{30}}{\delta x^1} \right] \frac{dr}{d\tau} \frac{dt}{d\tau} = - \frac{d^2\phi}{d\tau^2} \neq 0$$

That is to say: There is a tangential gravitational “force” on an object even if this object moves on a strictly anti-radial path. This can only be accounted for as follows: The tangential (**phi**-) components of the velocities of free falls of test bodies (all of which come from afar) changes along a radial line. Different from the Schwarzschild metric, the tangential components of those velocities are not zero in the Kerr metric. This again demonstrates that the gravitational “force” is not strictly anti-radial here.

cc) One might, of course, raise the question as to why $\mathbf{d}^2\mathbf{phi}/\mathbf{d}\tau^2$ vanishes in the Kerr metric if $\mathbf{d}\mathbf{r}/\mathbf{d}\tau$ is zero. The answer is: Due to reason of symmetry, the tangential components of velocities of free falls of test bodies (all of which come from afar) does not change along an arc, that is with varying

\mathbf{phi} and constant \mathbf{r} . The summands 3-30 and 3-03 in Equation (36), whose non-zero differential quotients are $d\mathbf{phi} dt/\tau^2$ and $dt d\mathbf{phi}/d\tau^2$, must therefore vanish.

In other words: Any tangential acceleration (which is due to “funny” space-time) experienced on a strictly circular orbit can only be the result of a resulting gravitational “force” vector whose direction forms an angle with a radial line of more than zero but less than 90° . This gravitational “force” is not revealed by the Christoffel symbol displayed in Equations (33) and (36) as long as $d\mathbf{r}/d\tau$ is set to zero.

3. Why the transverse “force” cannot be caused by the motion of a test mass across a “gravitomagnetic field”

In the past, it has been postulated that a tangential force, that is force at right angle to straight radial lines, does exist (in the vicinity of spinning masses) as a force analogous to the Lorentz force in electromagnetism. In perfect analogy to a magnetic field, the existence of a “gravitomagnetic field” has been postulated. It is thought of as being generated by the spinning of a heavy mass, analogous to the generating of a magnetic field by a spinning, electrically charged sphere. See, for instance, Ciufolini, Wheeler 1995 (Chapter 6.1, p. 316):

“In electrodynamics, in the frame in which an electrically charged sphere is at rest, we have an electric field. If we then rotate the sphere we observe a magnetic field, and the strength of this field depends on the angular velocity. Similarly, in geometrodynamics, a nonrotating, massive sphere produces the standard Schwarzschild field. If we then rotate the sphere, we have a gravitomagnetic field,”

Consequently, a Lorentz-like force has been postulated to occur when a test mass is moving at right angle with respect to these hypothetical field lines. This would result in a cycloidal motion of a test mass, similar to the $\mathbf{E} \times \mathbf{B}$ drift of an electric charge in electromagnetism [see Ciufolini, Wheeler 1995 (Chapter 1.1, Fig. 1.2 a, p. 8) for a depiction of the situation].

But Equation (38) is incompatible with such a force: If the anti-radial motion of a test mass is exchanged for radial motion of the same magnitude, the numerical values of the summands in the squared bracket have to be multiplied by -1. However, the numerical value of $d\mathbf{r}/d\tau$, too, has to be multiplied by -1. This is why the minus signs cancel each other. As a result, the “force” yielded by Equation (38) points in the same direction no matter whether the motion of the test mass is radial or anti-radial. But a Lorentz-like force WOULD change its direction (by 180°).

4. The sameness of temporal and spatial (radial) dilation/contraction factors as a necessary condition for the

qualification of a radial trajectory as a geodesic

When comparing the metric tensors (3) (Schwarzschild metric) and (18) (kerr metric) with each other (they both apply to radial displacements in the equatorial plane), we notice the following difference: In Equation (3) (Schwarzschild metric), the (squared) factor of time dilation of a stationary clock in the gravity field (\mathbf{g}_{00}) and the (squared) factor of length contraction of radially oriented, stationary meter stick ($1/\mathbf{g}_{11}$) are identical with each other, that is, “ $\mathbf{1-r}_s/\mathbf{r}$ ”. In Equation (18) (Kerr metric), the (squared) factor of time dilation of a stationary clock in the gravity field (\mathbf{g}_{00}) is the same as it is in the Schwarzschild metric; but the (squared) factor of length contraction ($1/\mathbf{g}_{11}$) is different, that is, “ $(\mathbf{1-r}_s/\mathbf{r}+\mathbf{a}^2/\mathbf{r}^2)$ ”. This can be accounted for as follows: As will be expanded below, both the dilation of time of a stationary clock and the contraction of radial length of a stationary meter stick (judged in the reference frame of a stationary observer outside of the gravity field) are the results of a flow of space that rushes past the clock and past the meter stick at a speed which is identical in absolute magnitude with the escape velocity at that location. As long as the flow of space is strictly anti-radial (toward the center of mass), the two factors are identical with each other. But in case the inward flow is not strictly anti-radial, but has a tangential component, things present themselves as follows: Presuming the overall speed of the flow is still the same as before when it passes by the stationary clock and the stationary meter stick, the factor of *time* dilation is unaffected. The stationary meter stick, however, cannot, when oriented in the \mathbf{r} -direction, contract in the \mathbf{r} -direction to the same extent as it did before. This is because the contraction in the \mathbf{r} -direction is now only a *component* of the total contraction, that is, the contraction in the direction of the gravity vector. Therefore the component of contraction in the \mathbf{r} -direction (expressed as a dimensionless factor) is not the same as the temporal dilation factor.

In order to get a deeper insight, we consider a stationary meter stick (of unit length) which is lying in the equatorial plane. It shall be changing its orientation from radial to tangential. Then its length (expressed as a dimensionless fraction of a meter) starts with $(\mathbf{1-r}_s/\mathbf{r} + \mathbf{a}^2/\mathbf{r}^2)^{1/2}$. Somewhere in between, it shrinks to $(\mathbf{1-r}_s/\mathbf{r})^{1/2}$. This is the orientation at which it is aligned with the gravitational vector and hence with the spatial component of a geodesic. Finally, when it has reached the tangential orientation, its length is bigger than unity, that is, $(\mathbf{1+a}^2/\mathbf{r}^2+\mathbf{a}^2\mathbf{r}_s/\mathbf{r}^3)^{1/2}$. Different from the Schwarzschild metric, tangentially oriented, stationary meter sticks in the equatorial plane are subject to a relativistic change in length. Their length is not reduced, but enlarged.

How is this accounted for? The answer can only be: It is a consequence of the concentration of gravitational field lines near the equator, which is a peculiarity of the Kerr metric (see Bicak, Stuchlik 1976, p. 384: “*The theta-*

dependence of the third term in ... implies that the particles moving in the equatorial plane fall faster than the particles along the axis of symmetry.) This concentration of field lines is caused by the fact that the number of stationary meter sticks – laid end to end – needed to form a closed circle in the equatorial plane is reduced when the spherical object is spinning.

Why is the radially oriented meter stick also affected by this phenomenon (and not just a tangentially oriented one)? In other words: Why does the g_{11} -component of the Kerr metric displayed in Equation (18) contain the parameter “ a ”? The answer is: The extension of length in the Kerr metric is a phenomenon which turns up at right angle with respect to gravitational field lines. Since the field lines are not strictly radial, a component of this length extension is also measurable in the strictly radial direction.

To summarize: A difference between the dilation factor of local time on the one hand and the contraction factor of local radial length on the other hand is a reliable indicator for radial motion being incapable of constituting a motion along a geodesic.

5. The ubiquitous (and not just local) transforming away of a gravitational “force” including its tangential component by flowing space

a) The dilation of time and the shortening of meter sticks as phenomena which cannot alone account for gravitational acceleration

We will now determine how our result of a non-conservative gravitational field around a spinning, spherical mass translates into a picture of moving space volumes. More precisely: We will replace the existing picture of space swirling around a spinning, gravitating body by a corrected one. This will enable us to determine the magnitude of a tangential gravitational “force” in the realm of the Kerr metric.

aa) Whenever a test object is gathering speed in a gravity field by increasing its kinetic energy, this is accounted for by the geometry of spacetime. That is to say: The object was simply following a geodesic. See I. Ciufolini, J.A. Wheeler (Ciufolini, Wheeler 1995)(Chapter 2.4, p. 27):

” According to the field equation, $G^{ab} = k T^{ab}$, mass–energy T^{ab} ‘tells’ geometry G^{ab} how to ‘curve’; furthermore, from the field equation itself, geometry ‘tells’ mass–energy how to move.”

Though being correct, this famous statement is incomplete: Geometry alone does not make mass-energy move; in addition, it needs an accelerating flow of space. Such a flow is implicitly contained in the Schwarzschild and in the Kerr metric.

bb) This shall be explained in greater detail. Consider a test object in free radial fall in the realm of the Schwarzschild metric. As we know, this test

object finds itself on a \mathbf{r} -geodesic. The equation of an \mathbf{r} -geodesic is:

(39)

$$\frac{d^2 R}{d\tau^2} = -\Gamma_{\mu\nu}^r \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

The right-hand side of the equation is different from zero. We know that “funny primed coordinates” – caused by “funny” spacetime – are responsible for this.

However, curvature of spacetime as such cannot alone account for an acceleration. In addition, space itself has to be in accelerating motion, so that it takes objects embedded in it along with the ride. In order to realize this, let us consider the following equation for the squared escape velocity derived from the Schwarzschild metric (see below for details):

(40)

$$\frac{v_{\text{escape}}^2}{c^2} = \frac{v_{\text{freefall}}^2}{c^2} = \frac{(v')_{\text{escape}}^2}{c^2} \left(1 - \frac{r_s}{r}\right)^2 = \frac{r_s}{r} \left(1 - \frac{r_s}{r}\right)^2$$

It is also the squared velocity of an object in free anti-radial fall. It shows that the velocity of an object in free fall, judged in the coordinate system of a distant, stationary observer outside of the gravity field, is almost zero at very large \mathbf{r} , and is also zero at $\mathbf{r}=\mathbf{r}_s$. But between these extremes, it does of course exceed zero. More precisely, we have for the maximum value of $\mathbf{v}_{\text{freefall}}$ [with $\mathbf{v}(\mathbf{r})_{\text{freefall}}$ being defined as numerically positive for $\mathbf{r} > \mathbf{r}_s$]:

(41)

$$\frac{1}{c} \frac{dv_{\text{freefall}}}{dr} = \frac{d\left[+\sqrt{\frac{r_s}{r}} \left(1 - \frac{r_s}{r}\right)\right]}{dr} = -\frac{\frac{r_s}{2} \left(r - 3r_s\right)}{+\sqrt{\frac{r_s}{r}} r^3} = 0$$

$$\Rightarrow r = 3r_s \Rightarrow v_{\text{freefall}}(3r_s) = v_{\text{max}} = 0.3849c$$

According to Equation (41), the test object is gathering speed in free anti-radial radial fall until it has arrived at the position $\mathbf{r}=\mathbf{3r}_s$, where its velocity has reached a maximum value (which is $0,3849c$, regardless of the numerical value of \mathbf{r}_s). From then on, its speed is declining to zero (at $\mathbf{r}=\mathbf{r}_s$). However, both the contraction of stationary, radially oriented meter sticks, and also the dilation of time of stationary clocks increase with diminishing \mathbf{r} . Hence, there should nowhere be an increase in velocity of a test object in free anti-radial fall (judged in the coordinate system of the distant observer), but a steady slowing down. In other words: Curvature of spacetime accounts for the strange behaviour of stationary clocks and stationary meter sticks in the gravity field, but doesn't give rise to the expectation of an *increase in speed of objects* coasting in the gravity field along an anti-radial path.

b) The emergence of space volume elements as a further condition for any gravitational acceleration experienced by a freely falling observer

aa) The only explanation – which does not require any real force – for an increase in velocity \mathbf{v} of the freely falling observer is the accelerating motion of space. This has been well known for many decades, but so far it has been believed that such a “transforming away” of gravity as a force cannot be done ubiquitously, but only locally. See, for instance H. Reichenbach (p. 226):

“Let us consider for example the radial field of the earth (...). If we let a rigid system of cells (...) Move in the direction of arrow b with an acceleration $g=981 \text{ cm/sec}^2$, the earth field will be transformed away in a cell but a but not in any of the others. We can now make the following statement: for any given small region b we can always specify for the system of cells an accelerated motion which will transform away the gravitational field at b . We may therefore say that any gravitational field can always be transformed away in any given region, but not in all regions at the same time by the same transformation.”

Contrary to that opinion, the accelerated motion of space CAN be conceived of as an ubiquitous (and not just local) phenomenon. How can this be achieved without inner contradictions? The answer is: Space volume elements emerge in space out of nothingness, and flow toward the spherical body in an ever faster, accelerating, anti-radial motion. They vanish into nothingness in the interior of the spherical mass.

One should keep in mind that the Schwarzschild solution comes in a cosmological variant, too (De-Sitter-universe), in which the universe is expanding (whereas the system of coordinates used does not, but is still an ordinary system of polar coordinates). Space in this model of the universe, too, is constantly (and undisputedly) emerging out of nothingness, in order to flow toward the cosmic event horizon in an ever faster motion. This steady and ubiquitous emerging of space volume elements is also responsible for the red-shift of light received from distant galaxies. What thus applies to *one* variant of the Schwarzschild solution, can be applied to the other variant (the one for spherical, non-spinning bodies) as well.

In short: The cosmic variant of the Schwarzschild solution can be said to describe a gravitational acceleration of space and objects embedded in it towards the *cosmic event horizon*; whereas the variant for a spherical mass describes a gravitational acceleration of space and objects embedded in it towards the *Schwarzschild horizon*.

cc) This can be made evident (for the spherical mass) by re-considering Equation (12), which gives the local gravitational acceleration. When forming the integral of Equation (12), we get:

$$(42) \quad -E'_{pot} = - \int_r^\infty g'(r)dr = - \int_r^\infty -\frac{1}{2} \frac{c^2 r_s}{r^2} dr = - \left[\frac{1}{2} \frac{c^2 r_s}{r} \right]_r^\infty =$$

$$\frac{1}{2} \frac{c^2 r_s}{r} = E'_{kin} = \frac{1}{2} (v')_{escape}^2$$

The term \mathbf{g}' is the local gravitational acceleration or the local gravitational “force” per unit mass (felt by a stationary observer in the gravity field who uses primed coordinates). The term $-\frac{1}{2} \mathbf{c}^2 \mathbf{r}_s / \mathbf{r}$ is the local potential energy \mathbf{E}'_{pot} per unit mass of the test body. It must be equal in magnitude to the local kinetic energy \mathbf{E}'_{kin} per unit mass of the test body if the test body has come from infinity.

We thus get for local escape velocity from Equation (42):

$$\frac{(v'_{escape})^2}{c^2} = \frac{(v'_{freefall})^2}{c^2} = \frac{r_s}{r}$$

It is the same as in Newtonian physics. At the Schwarzschild horizon \mathbf{r}_s (of a non-spinning spherical object), the local escape velocity is thus equal to \mathbf{c} .

[One could perhaps wonder why \mathbf{r} , i.e., circumference divided by 2 pi, and not \mathbf{R} , i.e., radial distance measured by radially oriented, stationary meter sticks laid end to end, is used as the independent variable in the integral which appears in Equation (42). The answer is: In case we had used \mathbf{R} and not \mathbf{r} , the escape velocity at $\mathbf{r}=\mathbf{r}_s$ would exceed \mathbf{c} . This is because the numerical value of $\mathbf{Delta R}$ (in meters) between two fixed points in space is larger than that of $\mathbf{Delta r}$, so that, as a consequence, the integral would be larger. But this cannot be physically correct.]

Moreover, Equation (43) can be turned into:

$$1 - \frac{(v'_{escape})^2}{c^2} = 1 - \frac{(v'_{freefall})^2}{c^2} = 1 - \frac{r_s}{r}$$

We can also formulate:

$$\frac{d\tau^2}{dt^2} = 1 - \frac{r_s}{r} = 1 - \frac{(v'_{escape})^2}{c^2} = 1 - \frac{(v'_{freefall})^2}{c^2} = \frac{(dt')^2}{dt^2}$$

The first equality sign in Equation (45) owes its presence to the Schwarzschild solution of Einstein’s field equation; the last equality sign owes its presence to the Lorentz transformation of space and time valid in Special Relativity for two reference frames (primed and unprimed) which are in straight and unaccelerated motion relative to each other.

From Equations (44) and (45) we can infer: According to the Schwarzschild metric [from which Equation (12) has been derived], a stationary observer, whose name is Alice and who is sitting in the gravity field of a spherical, non-spinning body, is subject to the same dilation of time (with respect to a

second observer, whose name is Bob, who sits outside of the gravity field) as would be the case if Alice were in straight and unaccelerated motion outside of any gravity field at a velocity (with respect to Bob) which is the same in magnitude as the escape velocity from Alice's position in the gravity field. In simpler words: The time dilation experienced by stationary Alice in the gravity field can be explained by the fact that space around Alice is rushing past her at a velocity which is identical in magnitude with the impact velocity of a meteorite that had come from far away and had a velocity of only a little more than zero when it entered the gravity field. That is to say: Both in Special Relativity and in the realm of the Schwarzschild metric, time dilation is the result of the *velocity* of an observer, either with respect to another observer (as is the case in Special Relativity), or with respect to space that is rushing past a stationary observer sitting in a gravity field.

cc) This easily explains the phenomenon of a difference in the quotients $d\tau^2/dt^2$ for two orbiters on a circular path around a spinning spherical mass in its equatorial plane, with orbital radius and squared angular velocity $d\phi^2/dt^2$ being the same for the two orbiters (by arrangement), but their directions of orbiting being opposed to each other. The equation of the Kerr metric for an orbit in the equatorial plane, gives:

(46)

$$\frac{d\tau^2}{dt^2} = \left(1 - \frac{r_s}{r}\right) - \frac{1}{c^2}(r^2 + a^2 + \frac{r_s a^2}{r}) \frac{d\phi^2}{dt^2} + \frac{2r_s a}{rc} \frac{d\phi}{dt}$$

The orbiter whose direction of orbiting coincides with the direction of the spin of the spherical mass (the factor \mathbf{a} and $d\phi/dt$ have the same sign then) has a lower velocity relative to the space which is rushing by, therefore the quotient $d\tau^2/dt^2$ (<1) which refers to that orbiter is larger (closer to unity) than that which refers to the other orbiter (with respect to which \mathbf{a} and $d\phi/dt$ differ in sign).

c) Accelerating flows of space in the cosmic variant of the Schwarzschild solution

aa) Finally, the cosmic variant of the Schwarzschild solution provides another confirmation of our recognition according to which both in Special and General Relativity time dilation (and also the shortening of meter sticks) is caused by the relative motion of space, and by nothing else.

For this purpose, it is worthwhile noticing the following relationship (derived from the cosmic variant of the Schwarzschild solution on the one hand, and from Special Relativity on the other hand):

(47)

$$\frac{d\tau^2}{dt^2} = 1 - \frac{H^2 r^2}{c^2} = 1 - \frac{(v'')^2}{c^2} = \frac{d\tau^2}{(dt'')^2} \Rightarrow dt^2 = (dt'')^2$$

In Equation (47), time $\mathbf{tau} = \mathbf{t}'$ is the time of a distant observer Alice who is stationary with respect to the Milky Way, \mathbf{v}'' is the velocity of Alice in the reference frame of a distant galaxy which is rushing past Alice; \mathbf{t}'' is the time measured in the reference frame of that galaxy. In the reference frame of the Milky Way (whose time is \mathbf{t}), time dilation of the clocks held by far-away Alice (who is stationary with respect to the Milky Way) is brought about by the motion of space past Alice, with the speed of that motion being $\mathbf{H}^2 \mathbf{r}^2$ when measured as local speed by Alice (\mathbf{H} is Hubble's constant, \mathbf{r} is the circumference of a circle with the Milky Way at its center, divided by 2π). In the double-primed reference frame of the distant galaxy rushing past Alice, it is the Lorentz transformation of Special Relativity which is applied to nearby Alice. Observers in that galaxy state the same dilation of Alice's time as observers in the Milky Way do. But regarding the time dilation stated by observers in the distant galaxy in their frame of reference, this dilation is caused not by the local speed of space, but by the local speed of Alice. It shows that it is the speed of space – in the reference frame of the Milky Way – what generates time dilation and a shortening of meter sticks in the realm of the cosmic Schwarzschild metric. But there is no reason to assume that things are different in the Schwarzschild metric for of a spherical mass.

bb) Moreover, this thought experiment reveals (as a side product) another important feature in Relativity, namely: $d\mathbf{t} = d\mathbf{t}''$. The sameness of time lapse rates in the Milky Way and in the distant galaxy, which we derived from the cosmic variant of the Schwarzschild solution, is an affirmation of the cosmological principle, as a consequence of which our Milky Way does not play a special role in the universe. One should note that the cosmological principle is not (explicitly) contained in the two axioms of Relativity.]

cc) Another remarkable consequence pertains to the relativity principle when it comes to superluminal escape velocities of galaxies. Because of (48)

$$v_{light_{radial}}^2 = c^2 \left(1 - \frac{H^2 r^2}{c^2}\right)^2$$

(derived from the cosmic version of the Schwarzschild solution) being applicable even for $\mathbf{H}^2 \mathbf{r}^2 = (\mathbf{v}'')^2 > c^2$, Equation (48) can be converted into: (49)

$$\frac{d\tau^2}{dt^2} = \left|1 - \frac{H^2 r^2}{c^2}\right| = \left|1 - \frac{(v'')^2}{c^2}\right| = \frac{d\tau^2}{(dt'')^2} \Rightarrow dt^2 = (dt'')^2$$

The equality of $d\mathbf{t}^2$ and $(d\mathbf{t}'')^2$ holds true even for escaping galaxies which are beyond the cosmic event horizon. This presents a perfect match with the cosmological relativity principle, according to which our Milky Way does not play a special role in the universe.

d) Testable consequences of the ubiquitous, accelerating flow of space near heavy masses; Einstein on superposing motions of space

aa) A testable consequence of the recognition that space is flowing in accelerating motions is the following: An electrically charged body or particle, when in free fall along a geodesic, should not emit electromagnetic radiation. But if the gravitational field were a force field-like the electric field, the particle would have to give off radiation.

Likewise, a massive test body in free fall along a geodesic should not increase its mass. Although this statement can be found in textbooks and in oral lectures given at universities, authors and lecturers do not realize that this requires the gravitational field to be constituted by accelerating flows of space.

bb) Each of the two experiments would, at the same time, corroborate A. Einstein's concept of moving spaces. It was A. Einstein (Einstein 1961) (Appendix V – supplemented in 1952 by Einstein –, pp. 138, 139) himself who, a few years prior to his death, mentioned the possibility of moving space volumes:

“When a smaller box is situated, relatively at rest, inside the hollow space of a larger box S , then the hollow space of s is a part of the hollow space of S , and the same ‘space’, which contains both of them, belongs to each of the boxes. When s is in motion with respect to S , however, the concept is less simple. One is then inclined to think that s encloses always the same space, but a variable part of the space S . It then becomes necessary to apportion to each box its particular space, not thought of as bounded, and to assume that these two spaces are in motion with respect to each other. [] Before one has become aware of this complication, space appears as an unbounded medium or container in which material objects swim around. But it must now be remembered that there is an infinite number of spaces, which are in motion with respect to each other. The concept of space as something existing objectively and independent of things belongs to pre-scientific thought, but not so the idea of the existence of an infinite number of spaces in motion relatively to each other. This latter idea is indeed logically unavoidable, but is far from having played a considerable role even in scientific thought.”

Einstein's reflections are not restricted to cases of overlapping motions of space as blocks. They rather allow for a space whose velocity varies from place to place even in a Cartesian system of coordinates. In case a freely falling electric charge does not give off radiation, this concept – and hence Einstein's general view on moving space – is physically corroborated, since there is no way to account for the absence of radiation other than by an accelerating flow of space.

cc) In discussions on “frame-dragging” in the vicinity of spinning spherical masses, the notion of “flowing space” has been accepted by all participants in

the past, and hasn't met rejection in principle. Then these participants should be well prepared to proceed with an improved conception of flowing space.

e) An accelerating, stationary flow of space near a gravitating mass as an inevitable consequence of proper time being a Lorentz invariant in Relativity

aa) The same result is also reached from a different starting point. A central point in Relativity is the following: The interval of proper time $d\mathbf{tau}$ of an observer (who uses primed coordinates in his local frame of reference, so that $d\mathbf{tau}=d\mathbf{t}'$ by convention) between two events (happening at the same spatial location for that observer) is an invariant *spacetime* interval in other frames of reference. The invariance of proper time is a mathematical consequence of the two axioms of Relativity, i.e., the invariance of the local speed of light, and the relativity principle. When applied to a gravity field of a non-spinning spherical mass, this principle reads: The (squared) local time interval $d\mathbf{tau}^2$ between two local events happening at the same location in the gravity field, measured by a local, stationary observer in that gravity field (Alice), is equal to $d\mathbf{t}^2$ minus some squared spatial displacement, with both $d\mathbf{t}$ and the spatial displacement measured in the reference frame of a stationary observer who sits outside of the gravity field (Bob). But what spatial displacement can be meant thereby?

bb) In order to find an answer, we resort to the Schwarzschild metric. The Schwarzschild solution of Einstein's field equation gives for a stationary observer (whose time is $\mathbf{tau}=\mathbf{t}'$) in the gravity field:

(50)

$$\begin{aligned} d\tau^2 &= (dt')^2 = \left(1 - \frac{r_s}{r}\right) dt^2 = \left[1 - \frac{(v')^2_{freefall}}{c^2}\right] dt^2 = \left[1 - \frac{(ds')^2}{c^2 d\tau^2}\right] dt^2 \\ &= dt^2 - \frac{(ds')^2}{c^2(1-\frac{r_s}{r})} = dt^2 - \frac{(ds')^2}{c^2(1-\frac{r_s}{r})} \frac{ds^2}{(ds')^2(1-\frac{r_s}{r})} = dt^2 - \frac{ds^2}{[c(1-\frac{r_s}{r})]^2} = dt^2 - \frac{ds^2}{v_{light}^2} \end{aligned}$$

The time interval $d\mathbf{t}$ [= $d\mathbf{tau}/(\mathbf{1}-\mathbf{r}_s/\mathbf{r})^{\frac{1}{2}}$], measured by the observer outside of the gravity field, in Equation (50) shall be the time interval which elapses between two ticks of a stationary clock positioned next to the stationary observer in the gravity field. The term " $ds^2/[(ds')^2(\mathbf{1}-\mathbf{r}_s/\mathbf{r})]$ " amounts to unity (as will be shown below). The term ds' stands for the radial distance covered by the inward flow of space during the time interval $d\mathbf{tau}$, with the latter being the local time interval between two ticks of the stationary clock mentioned above. The flow of space is accelerating, as its speed \mathbf{v}'_{space} is equal to that of a freely falling observer ($\mathbf{v}'_{freefall}$).

When considering the very left and the very right-hand side of Equation (50), we realize that the requirement of an invariant spacetime interval is

met: A squared interval of proper time (whose end points are the temporal coordinates of two point events occurring at the same spatial location) in one frame of reference is equal to the difference between a squared temporal and a squared spatial interval (divided by the squared speed of light) in another frame of reference.

Different from flat spacetime (Minkowski spacetime) where the speed of light is \mathbf{c} everywhere, the squared spatial interval \mathbf{ds}^2 is not divided by the second power of \mathbf{c} , but by the second power of what the radial speed of light is near the central spherical mass in the (unprimed) reference frame of the stationary observer outside of the gravity field. There the (radial) velocity of light is lower than \mathbf{c} , i.e., $\mathbf{c}(\mathbf{1}-\mathbf{r}_s/\mathbf{r})$, as is shown below [in Equation (54)].

If expressed in that way, the very left and the very right-hand side of (50) are indistinguishable from the respective sides of the Minkowski metric. This is because the Minkowski metric can be formulated as follows:

(51)

$$d\tau^2 = dt^2 - \frac{dr^2}{v_{light}^2}$$

In this form, the Minkowski metric is indistinguishable from Equation (50) and hence from the Schwarzschild metric.

cc) It shall now be shown why the term “ $\mathbf{ds}^2/[(\mathbf{ds}')^2(\mathbf{1}-\mathbf{r}_s/\mathbf{r})]$ ” appearing in Equation (50) amounts to unity if \mathbf{ds}' (or \mathbf{ds} , respectively) is the radial distance between two stationary spatial points in the gravity field. Setting the magnitude of this term to unity is equivalent to saying that a radially oriented, stationary meter stick is shortened by the factor $(\mathbf{1}-\mathbf{r}_s/\mathbf{r})^{1/2}$ when looked at in the reference frame of a distant observer.

A proof of this assertion is provided when the speed of light in a gravity field is determined in the unprimed reference frame of a distant observer. Taking the Schwarzschild solution as a starting-point and setting the proper time $\mathbf{d}\mathbf{tau}$ of a light pulse to zero, we get for the *radial* speed of light:

(54)

$$0 = \left(1 - \frac{r_s}{r}\right) dt^2 - \frac{1}{c^2 \left(1 - \frac{r_s}{r}\right)} dr^2 \Rightarrow v_{light_{radial}} = \frac{dr}{dt} = c \left(1 - \frac{r_s}{r}\right)$$

In contrast, the *tangential* speed of light in the equatorial plane is [see Equation (4) for the Schwarzschild metric]:

(55)

$$0 = \left(1 - \frac{r_s}{r}\right) dt^2 - \frac{r^2}{c^2} d\phi^2 \Rightarrow v_{light_{tangential}} = \frac{rd\phi}{dt} = c \sqrt{\left(1 - \frac{r_s}{r}\right)}$$

The only explanation for the difference between Equation (54) and (55) is: While the tangential speed of light is reduced by the dilation of time (experienced by stationary clocks in the gravity field), the *radial* speed of light

is reduced both by the dilation of time and by the contraction of stationary, radially oriented meter sticks. This is the proof we have been looking for.

6. Determining the magnitude of the tangential gravitational “force” near a spinning mass by an analysis of the Kerr metric

a) We will now derive an equation of the tangential gravitational “force” in the equatorial plane of a spinning, spherical mass.

The equation of the Kerr metric for a circular orbit in the equatorial plane is [see Rezolla, O. Zanotti 2013 (Chapter 1.7.2, Eq. 1.251, p. 55)]:

$$(56) \quad d\tau^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \frac{1}{c^2} \left(r^2 + a^2 + \frac{r_s a^2}{r}\right) d\phi^2 + \frac{2r_s a}{rc} d\phi dt$$

or

$$(57) \quad \frac{d\tau^2}{dt^2} = \left(1 - \frac{r_s}{r}\right) \frac{dt^2}{dt^2} - \frac{1}{c^2} \left(r^2 + a^2 + \frac{r_s a^2}{r}\right) \frac{d\phi^2}{dt^2} + \frac{2r_s a}{rc} \frac{d\phi dt}{dt^2} \leq 1$$

In the foregoing, we found that time dilation is brought about the relative motion of space with respect to a clock; the higher the velocity of that relative motion, the larger is the time dilation, and the smaller is the quotient $d\tau^2/dt^2$ (ranging from almost zero to unity).

The velocity of flowing space has an anti-radial component $\mathbf{v}_{\text{radial}}$ and a tangential component \mathbf{v}_{tang} . The relative speed of an orbiting observer (who is orbiting in the direction of the spin of the spherical mass) with respect to space flowing past him or her is at its minimum – and the numerical value of the squared quotient expressing time dilation of the orbiter (relative to a distant observer sitting outside of the gravity field) is at its maximum – when, in the reference frame of a stationary observer outside of the gravity field, the tangential velocity of the orbiter is identical in magnitude and direction with the tangential component \mathbf{v}_{tang} of the velocity of space. In order to determine the magnitude of that tangential component of the velocity of space \mathbf{v}_{tang} , we simply have to differentiate Equation (57), that is, the squared factor of time dilation of the orbiter, by the angular velocity, and set the result to zero. We rename the angular velocity $d\phi/dt$ to $\boldsymbol{\omega}$, and then get from Equation (57):

$$(58) \quad \frac{d}{d\omega} \frac{d\tau^2}{dt^2} = \frac{d}{d\omega} \left[\left(1 - \frac{r_s}{r}\right) - \frac{1}{c^2} \left(r^2 + a^2 + \frac{r_s a^2}{r}\right) \omega^2 + \frac{2r_s a}{rc} \omega \right] = -\frac{2}{c^2} \left(r^2 + a^2 + \frac{r_s a^2}{r}\right) \omega + \frac{2r_s a}{rc} = 0$$

From Equation (58) follows:

(59)

$$\omega = \frac{r_s a c}{r (r^2 + a^2 + \frac{r_s a^2}{r})}$$

The second derivative of $d\tau^2/dt^2$ with respect to ω is numerically negative:

(60)

$$\frac{d^2 \frac{d\tau^2}{dt^2}}{d\omega^2} = - \frac{2}{c^2} (r^2 + a^2 + \frac{r_s a^2}{r})$$

Hence the function $f(\omega) = d\tau^2/dt^2$ has a maximum – and not a minimum – at this value of ω (as expected).

Equation (59) can be converted into:

(61)

$$v_{tang} = \omega r = \frac{r_s a c}{r^2 + a^2 + \frac{r_s a^2}{r}}$$

With $c = 299.792 \times 10^6$ m/sec, $r = 6500 \times 10^3$ m; $a = J (Mc)^{-1} = 5.86 \times 10^{33}$ kg m²/sec $\times (5.97 \times 10^{24}$ kg $\times 299.792 \times 10^6$ m/sec)⁻¹ = 3.27 m, $r_s = 2GM/c^2 = 0.00887$ m, we get for an orbit around Earth at a distance r of 6500 km from the center: $v_{tang} = 0.20 \times 10^{-6}$ m/sec.

For the angle α between a radial line and a resulting gravitational field line [see Equation (40) for v_{escape}] we get (if $g_{radial} \gg g_{tan}$, with g being gravitational acceleration):

(62)

$$\tan \alpha = \frac{g_{tang}}{g_{radial}} = \frac{v_{tang}}{v_{radial}} \approx \frac{v_{tang}}{v_{escape}} = \frac{r_s a}{(r^2 + a^2 + \frac{r_s a^2}{r}) (1 - \frac{r_s}{r}) \sqrt{\frac{r_s}{r}}} = 0.1809 \cdot 10^{-10}$$

Since the directions of the flow of space and of gravity coincide with each other, and since both parameters are vectors, we have $g_{tan}/g_{radial} = v_{tang}/v_{radial}$ in Equation (62).

From Equation (62) follows:

(63)

$$\alpha = \arctan 0.1809 \cdot 10^{-10} = 0.18 \cdot 10^{-10} \text{ rad} = 0.18 \cdot 10^{-10} \cdot \frac{360^\circ}{2\pi} = 1.03 \cdot 10^{-9}^\circ$$

In case of $g_{radial} \gg g_{tan}$, the tangential gravitational acceleration g_{tan} is [according to Equation (62)]:

(64)

$$g_{\text{tan}} \approx g_{\text{radial}} \frac{r_s a}{(r^2 + a^2 + \frac{r_s a^2}{r}) (1 - \frac{r_s}{r}) \sqrt{\frac{r_s}{r}}} = g_{\text{radial}} \cdot 0.1809 \cdot 10^{-10} = 1.7 \cdot 10^{-10} \text{ m/sec}^2$$

In case the near-earth position is replaced by a position near a super-massive black hole, the numerical value of \mathbf{g}_{tan} is much greater.

7. How the rotational energy of the spinning mass is reduced

The rotational kinetic energy of the spinning black hole should be assumed to be reduced to the same extent to which the tangential component of its non-conservative gravitational field generates kinetic energy of accelerated particles (or larger objects) in its vicinity. This is because the gravitational field lines of objects orbiting the fast spinning mass (which can be a black hole), too, are not straight, but are slightly curved due to the distortion of spacetime caused by the central, spinning mass. The orbiters' field lines thus result in a torque exerted on that central mass, even if the density of the central mass is radially symmetrical (in a horizontal plane).

One should note that there doesn't exist a good analogy with electromagnetism, especially with the effect of counter-induction. The slowing down of the spin rate of the central mass takes place even if an orbiter travels at constant tangential speed. This is the case if frictional forces (resulting from gas particles or other obstacles) which is has to overcome prevent the orbiter from gathering speed despite the permanent push that it is receiving by the tangential component of the gravitational field. By contrast, counter-induction in electromagnetism requires an *accelerating* flow of orbiting (charged) particles.

8. Summary of results

The following results can be short-listed:

- The gravitational field lines of a spinning, massive sphere are not strictly straight, but are, in the equatorial plane, curved with a tangential component.
- The gravitational field around a spinning spherical body is therefore not conservative. Instead, an orbiter held on any circular orbit in the equatorial plane (outside of the radius which is equivalent to the Schwarzschild radius of non-spinning spheres) experiences a steady onward force like a charged particle in a ring-shaped particle accelerator does.
- The steady absorption of work by the orbiter is at the expense of the rotational kinetic energy of the central, spinning mass.
- A “gravitomagnetic field” is incompatible with the Kerr metric. For if it existed, a gravitational Lorentz force, too, would have to exist. But it is shown that such a force is absent within the Kerr metric.

Appendix: Small inverted arrows of time in the Schwarzschild and Kerr metric

a) In the foregoing, we came across the following relationships. On the one hand, we obtained

(65)

$$\begin{aligned}\Gamma_{00}^r \frac{dt^2}{d\tau^2} &= \left[\frac{g^{11}}{2} \left(\frac{\delta g_{10}}{\delta x^0} + \frac{\delta g_{10}}{\delta x^0} - \frac{\delta g_{00}}{\delta x^1} \right) \right] \frac{dt^2}{d\tau^2} \\ &= -\frac{c^2}{2} \left(1 - \frac{r_s}{r} \right) \frac{-d(1-r_s/r)}{dr} \frac{dt^2}{d\tau^2} = \left(1 - \frac{r_s}{r} \right) \frac{c^2 r_s}{2r^2} \frac{dt^2}{d\tau^2}\end{aligned}$$

for the Schwarzschild metric, and

(66)

$$\begin{aligned}\Gamma_{00}^r \frac{dt^2}{d\tau^2} &= \left[\frac{g^{10}}{2} (\dots) + \frac{g^{11}}{2} \left(\frac{\delta g_{10}}{\dots} + \frac{\delta g_{10}}{\dots} - \frac{\delta g_{00}}{\delta x_1} \right) \right] \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} \\ &= \frac{c^2}{2} \left(1 - \frac{r_s}{r} + \frac{a^2}{r^2} \right) \frac{d(1-r_s/r)}{dr} \frac{dt^2}{d\tau^2} = \left(1 - \frac{r_s}{r} + \frac{a^2}{r^2} \right) \frac{c^2 r_s}{2r^2} \frac{dt^2}{d\tau^2}\end{aligned}$$

for the Kerr metric. Both results are non-zero ones. They are expressions of the change in radial or anti-radial velocity of space with time at a fixed location in space. In other words: They describe the behaviour of a test object that has been thrown upward and is now finding itself at its turning point (apex) where it is standing still for a moment.

On the other hand, we obtained (for the Kerr metric):

(67)

$$\Gamma_{00}^\phi \frac{dt^2}{d\tau^2} = \left[\frac{g^{30}}{2} \left(\frac{\delta g_{00}}{\delta x^0} + \frac{\delta g_{00}}{\delta x^0} - \frac{\delta g_{00}}{\delta x^0} \right) + \frac{g^{33}}{2} \left(\frac{\delta g_{30}}{\delta x^0} + \frac{\delta g_{30}}{\delta x^0} - \frac{\delta g_{00}}{\delta x^3} \right) \right] \frac{dt^2}{d\tau^2} = 0$$

One should realize that it is the *non*-zero result of the **r**-00 summand in the Schwarzschild and Kerr metric which is puzzling, and not the zero result of the **phi**-00 summand in the Kerr metric. Given the gravitational field is static, any displacement in time alone, that is, any -00 summand, should be zero.

b) aa) The puzzle is solved by means of the theorem of reversibility of all processes in Newtonian and relativistic physics. According to this theorem, a sequence of events captured on a video clip shows physically possible events even if the clip is presented in reverse. So what about a video clip that has captured the free radial fall of a test body in the gravitational field of a non-spinning planet? If shown in reverse, it does *not* show a situation in which the mutual, attractive force between the planet and the test body is changed into a repulsive force. A repulsive force would entail that the velocity of the test

object fleeing from the planet would increase. But this is not what the video clip shows. It depicts a test body which behaves as if it had been tossed up, with its upward motion slowing down. In terms of flows of space, the video clip shows a radial, outward flow of space that leaves the interior of the planet at a speed identical with the escape velocity from the surface. The flow is slowing down with increasing distance from Earth. The test object is embedded in that flow. Its speed relative to the space element that surrounds it – which may be zero or non-zero – is staying constant.

Given the reversibility principle is physically correct, the decelerating, outward flow of space must be a physical reality. But this flow becomes physical reality only in case there is an object which is thrown upward. It is absent in all other situations. The appearing of a radially reversed, that is, outward flow of space must be rated as a partial time reversal, that is, as a small inverted arrow of time. In case of a momentary standstill of a test object at the apex of its trajectory, the vanishing of the outward flow of space and its replacement by an inward flow (with the two flows overlapping and thus offsetting each other for a period of time) explains why the radial component of the velocity of space changes with time even though the gravitational field is static: The partial time reversal comes to an end. It is as if the direction of the flow of space changed from outward to inward with time τ , quite similar to the situation of an electric circuit containing a condenser and a coil at the moment the current has come to a standstill and is about to begin flowing in the opposite direction. At this moment, the current strength and the magnetic flux generated by it are zero, but the derivative (of the current strength or the magnetic flux) with respect to time is not. In the gravitational case, the velocity of the flow of space is zero, because the two opposing flows of space are overlapping and are exactly cancelling each other at that moment in time. The downward flow prevails from then on, whereas it had been the upward flow which had prevailed prior to that moment.

bb) As regards the *tangential* component of the flow of space in case the spherical, gravitating mass is spinning, a video clip presented in reverse would show a change the direction of that flow from clockwise to anti-clockwise, or vice versa. But it would also show a change in direction of the spin of the spherical, gravitating mass. That is to say: There is no need to assume that, for one and the same sense of spinning of a gravitating mass, the tangential component of the flow of space comes in two opposing directions. This is why the tangential (ϕ -) component of the velocity of space at a fixed point in space does not change with time.

c) It is hardly known that it was L. Boltzmann who mentioned the possibility of opposing arrows of time more than a century ago. Though his main focus was on regions of the universe with opposing arrows of time separated from each other by huge distances (so that no overlapping of opposing arrows

of time would occur), he quite obviously also conceived of opposing arrows of time that would *overlap* [L. Boltzmann (1995), §90, pp. 447, 448]:

“For the universe, the two directions [of time] are indistinguishable, just as in space there is no up and down. ... Very well, you may smile at this; but you must admit that the model of the world developed here is at least a possible one, free of inner contradictions, and also a useful one, since it provides us with many new viewpoints. It also gives an incentive, not to speculation, but also to experiments (for example on ... the resulting deviations from the equations of hydrodynamics, diffusion, and heat conduction) which are not stimulated by any other theory.”

Boltzmann’s reflections were of course based on his understanding of entropy, and not of Relativity. Nevertheless, they reveal that it is not the two opposing arrows of time in the same region of space which require an explanation, but it is the existence of one single arrow of time only which does so. Then it should be no surprise to realize that the description given by General Relativity on the throwing up of a ball features two opposing arrows of time, namely, a big one and a small one.

Conflict of interest. The author declares no conflict of interest.

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