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What the Covariant and Ordinary Divergences of the Tensors in Einstein's Field Equation Tell about Newton's Apple When it Hits the Ground

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Abstract

The covariant divergence of each tensor in Einstein's field equation is zero as a mathematical necessity (and hence as a tacit presupposition), whereas the ordinary divergence is not necessarily so. The principle of energy conservation is thus observed in Einstein's field equation, but only if all kinetic energies obtained by gravitational accelerations (and hence by "forces" which are no *real* forces) are thrown out. This leads to an apparent dilemma for the principle of energy conservation when Newton's apple hits the ground, where the kinetic energy generated by gravitational acceleration converts into thermal energy and can thus no longer be disregarded. The dilemma can be solved. The solution sheds light on the disputed hypothesis according to which the gravitational field does not carry any energy at all. It also provides a surprising insight into the nature of dark energy – not as a result of speculations, but as a mathematical consequence of the covariant divergence of all tensors in Einstein's field equation being zero. In addition, the solution sheds light on the disputed concept of the world as a spatially four-(or more) dimensional brane.

Keywords: Einstein's field equation, energy-momentum tensor, divergence, energy of the gravitational field, dark energy, brane universe, fourth spatial dimension

Introduction

Few solutions of Einstein's field equation have been found so far. Nevertheless, there are some general conclusions (valid in any metric) that can be derived from Einstein's field equation. A key to some of those conclusions is a scrutiny of the covariant divergence of the tensors appearing in it.

1) The zero-value of the covariant divergence of the tensor T
Einstein's field equation reads:
(1)

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \lambda g^{\mu\nu} = 8\pi G T^{\mu\nu}$$

R with superscript denotes the Ricci tensor, which is an expression of the curvature of spacetime; **g** with superscript is the metric tensor (see above) which can be considered as being another expression of the curvature of spacetime. **R** without superscript is the contracted Ricci tensor (scalar). **Lambda** is Einstein's cosmological constant. **G** is Newton's gravitational constant. **T** with superscript is the energy-momentum tensor. As a mathematical necessity, the *covariant* divergence of both sides is zero, but the *ordinary* divergence can be different from zero [see Ciufolini, Wheeler (1995), Chapter 2.4, Eq. 2.4.2, p. 28]. The zero-value of the covariant divergence of any of the two sides of Einstein's field equation, that is, the correctness of
(2)

$$\nabla_\mu G^{\mu\nu} = \nabla_\mu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0$$

follows apriori without any reference to experiments [**lambda** is set to zero for a moment and therefore does not appear in Equation (2)]. For we can formulate:
(3)

$$\nabla_\mu R^{\mu\nu} = \frac{1}{2} g^{\mu\nu} \delta_\mu R$$

and can thus replace the covariant divergence of the **R**-tensor by an expression which contains the ordinary divergence of the contracted Ricci-tensor (that is, the gradient of the Ricci-scalar).

Moreover, since the covariant divergence of any metric tensor **g** is zero, and since the covariant divergence of the contracted Ricci-tensor is equal to its ordinary divergence (gradient of the Ricci-scalar), we can formulate (using the chain rule of differentiation):
(4)

$$\nabla_\mu \frac{1}{2} g^{\mu\nu} R = \frac{1}{2} (\nabla_\mu g^{\mu\nu}) R + \frac{1}{2} g^{\mu\nu} \nabla_\mu R = \frac{1}{2} g^{\mu\nu} \nabla_\mu R = \frac{1}{2} g^{\mu\nu} \delta_\mu R$$

With the left sides of (3) and (4) being equal to each other, the covariant divergence of the bracket in (2) must vanish, and (2) is thereby proved to be correct.

The result is not changed when the cosmological constant **lambda** – appearing in (1) – is different from zero: The covariant divergence of the metric tensor **g** is always zero; a scalar constant **lambda** in front of the tensor **g** cannot have any effect on this outcome.

With the covariant divergence of the left-hand side of Equation (1) being zero, the covariant divergence of the right-hand side of Equation (1) must also be zero. That is to say:
(5)

$$0 = \nabla_\mu T^{\mu\nu} = \delta_\mu T^{\mu\nu} + \Gamma^\mu_{\alpha\mu} T^{\alpha\nu} + \Gamma^\nu_{\alpha\mu} T^{\mu\alpha}$$

Quod erat demonstrandum. [The meaning of a zero result in (5) is a four-dimensional vector whose components are all zero.]

2) The ordinary divergence of the tensor T in a Cartesian frame of reference: zero in flat spacetime, and different from zero in a gravitational field

a) At this point, one shall be reminded of what the difference between an ordinary divergence and the covariant divergence is all about:

We shall distinguish between "funny" and "unfunny" systems of coordinates (to use an expression of L. Susskind's). A "funny" system of coordinates is a system in which the coordinate axes and their parallels intersect at angles different from 90°, or in which, the basic vectors (of unit length) change in length from place to place when compared to unit vectors of an overlying, "unfunny" coordinate system in which all coordinate axes and their parallels intersect at right angles.

Whenever the ordinary divergence of a vector in any "funny", non-rectangular coordinate system differs from zero while the covariant divergence (for the determination of which the use of a second and "unfunny", i.e., rectangular system of coordinates is necessary) of the same vector does not, it is so only because of the use of those "funny coordinates". See Fleisch (2012), Chapter 5.8, p. 153:

"*With Christoffel symbols in hand, you have a way of differentiating a vector or higher−order tensor that includes the effect of changes (if any) in the magnitude and direction of the basis vectors used to expand that vector or tensor. This type of derivative is called the 'covariant' derivative*, ...".

Those "funny" coordinates are, in turn, either the result of our free choice, or of "funny spacetime".

b) Thanks to Equation (5), we can even be more precise, and are in a position to distinguish between the two alternatives, i.e., whether or not "funny" spacetime is to blame for the non-zero ordinary divergence of the tensor **T** in the "funny", non-rectangular system of coordinates: Let us use two overlapping systems of coordinates, a primed one with "funny", non-rectangular coordinates and an unprimed one with "unfunny", rectangular coordinates (e.g. Cartesian). The two coordinate systems shall be stationary with respect to each other. For the determination of the covariant divergence of the tensor **T**, the use of the unprimed, "unfunny" (rectangular) coordinates and the knowledge how the primed basic vectors change relative to the unprimed ones is necessary.

The reason why one is able to distinguish between the two said alternatives is the following: If it were not "funny" spacetime, but only the choice of weird coordinates which is to blame for the non-zero ordinary divergence of the tensor **T** in the "funny", non-rectangular system of coordinates, the ordinary divergence of the tensor **T** in the unprimed ("unfunny"), rectangular system of coordinates would have to be zero. This is because in flat spacetime (and *only* there), the ordinary and the covariant divergence are indistinguishable from each other if "unfunny", rectangular coordinates are used. Given the covariant divergence of the tensor **T** is zero according to Equation (2), it follows that the ordinary divergence of the tensor **T** MUST be zero in the unprimed, rectangular system of coordinates in case spacetime is flat. Conversely, it follows that spacetime must be "unflat", i.e., curved, in case the ordinary divergence of the tensor **T** is different from zero in the unprimed, rectangular system of coordinates.

Recapitulating, we are thus confronted with the following situations:

— In case the ordinary divergence of the tensor **T** is zero in the unprimed, rectangular system of coordinates while the covariant divergence of the tensor **T** is also zero, we are sure that the non-zero ordinary divergence of the tensor **T** in the primed, non-rectangular system of coordinates is simply the result of our choice of "funny" primed coordinates, and not the result of funny spacetime.

– In case the ordinary divergence of the tensor **T** is different from zero in the unprimed, rectangular system of coordinates whereas its covariant divergence is zero, we are sure that it is "funny" spacetime which is to blame for the outcome and hence for the "funny" primed (non-rectangular) coordinates.

– In mathematical terms, the former case, that is, flat spacetime, is spelled out as follows (**delta**$_{\mathrm{ij}}$ is the Kronecker Delta, that is, a tensor whose off-diagonal components are all zero, and whose diagonal components are all equal to unity; **g** is the metric tensor):

(6)

$$\nabla_\mu T^{\mu\nu} = 0 \;\wedge\; \delta_{\mu'} T'^{\mu'\nu'} \neq 0 \;\wedge\; \delta_\mu T^{\mu\nu} = 0 \;\rightarrow\; g^{\mu\nu} = \delta_{ij}$$

If spacetime is "unflat", i.e., curved, we have:

(7)

$$\nabla_\mu T^{\mu\nu} = 0 \wedge \delta_\mu T^{\mu\nu} \neq 0 \rightarrow g^{\mu\nu} \neq \delta_{ij}$$

c) One must therefore clearly distinguish between two sorts of *reference frames* (those with "unfunny", rectangular, and those with "funny", non-rectangular coordinates) on the one hand, and two categories of *spacetime* (flat and "unflat", i.e., curved) on the other hand.

3) The zero-value of the ordinary divergence of the tensor T (in a reference frame with rectangular coordinates) as a necessary and sufficient condition for the local conservation of energy in general

a) Nevertheless, the principle of energy conservation requires that the ordinary divergence of the tensor **T** (that appears on the right-hand side of Einstein's field equation) is zero in a closed system if rectangular coordinates (e.g., coordinates in a Cartesian frame of reference) are used. This shall be illustrated:

In a four-dimensional Cartesian system of coordinates, the **T**-tensor can be represented by arrows which stand for vectors the tensor is made up of (four at most). More precisely: Each row (or column) of the 4 x 4 matrix representing **T** gives the four coordinates of a vector in four-dimensional diagram space. (The ordinary divergence of the tensor **T** is a vector with four components, with each component being given by the ordinary divergence of one of the four row- or column-vectors.) One of the vectors has the components $\mathbf{T}^{00}$, $\mathbf{T}^{01}$, $\mathbf{T}^{02}$, $\mathbf{T}^{03}$. In the Schwarzschild metric, only the component $\mathbf{T}^{00}$ is different from zero when it comes to representing the central, gravitating mass. Let the axis which represents the 0-component, that is, the time (**t**-) component of the four-dimensional vector, be the vertical one in our Cartesian system of coordinates. The vertically oriented vector $\mathbf{T}^{00}$**, 0, 0, 0** is thus an expression of the energy density (or mass density) **rho** of a volume element in space.

However, the volume element we will now scrutinize shall not be a volume element of the central mass, but shall be a volume element of a small test body. (As will be shown below, the *potential* energy of a mass or energy element is not included here.) Let the test body fall along an anti-radial path which coincides with one spatial coordinate axis ($\mathbf{x}^1$-coordinate). The volume elements of the test body are represented by a **T**-tensor which can be considered as being composed of two vectors only (rather than four). The four components of the first vector (representing the test body) are $\mathbf{T}^{00}$, $\mathbf{T}^{01}$, 0, 0. The four components of the second vector are $\mathbf{T}^{10}$, 0, 0, 0. This is because the T-tensor is symmetrical. Given there is a non-vanishing tensor component $\mathbf{T}^{01}$, there must thus also be a non-vanishing tensor component $\mathbf{T}^{10}$. The $\mathbf{T}^{10}$, 0, 0, 0 vector points straight in the vertical direction. However, since that vector is an expression of momentum and not of energy, we can ignore it, and may confine our attention to the $\mathbf{T}^{00}$, $\mathbf{T}^{01}$, 0, 0 vector.

The *magnitude* of the $\mathbf{T}^{00}$, $\mathbf{T}^{01}$, 0, 0 vector belonging to the test body can be represented in the diagram by a bundle of field lines, similar to the way the magnitude of an electric or magnetic field is expressed by field lines. Hence, there exists a bundle of field lines the number of which is an expression not of the energy *density*, but of the energy *contents* of the test body. If the mass/energy does not move in space over time ($\mathbf{T}^{01}$=0), all field lines of the bundle are strictly vertical. If the energy/mass *does* move in space (as is the case with our test body), the field lines are not strictly vertical, but have other components, too. In case the test body is in free anti-radial fall along a coordinate axis $\mathbf{x}^1 = \mathbf{r}$ (toward the central mass), the field lines have a horizontal component in the coordinate system, which is an expression of the momentum of the test body. That is to say: The vector component $\mathbf{T}^{01}$ then differs from zero. The *vertical* flux (number of field lines penetrating a horizontal plane in the coordinate system) is not affected thereby.

In order for the principle of energy conservation to be observed, the number of field lines (representing the $\mathbf{T}^{00}$, $\mathbf{T}^{01}$, 0, 0 vector field) which penetrate any horizontal plane (whose normal is vertical) must be constant. More precisely: Every field line must be free from disruption. But this is the same as saying that the divergence of the vector $\mathbf{T}^{00}$, $\mathbf{T}^{01}$, $\mathbf{T}^{02}$, $\mathbf{T}^{03}$ in our Cartesian system of coordinates must be zero.

With respect to the tensor $\mathbf{T}$, we can thus set up the very general statement (for any metric of spacetime): The principle of energy conservation is surely observed in case the ordinary divergence of the tensor $\mathbf{T}$, that is, the divergence of all row- or column-vectors $\mathbf{T}$ is made up of, is zero in a reference frame with rectangular coordinates (like that of a Cartesian frame of reference). If, on the contrary, the ordinary divergence of the tensor $\mathbf{T}$ in such a frame is different from zero (that is, if the ordinary divergence of at least one of the four row- or column-vectors does not vanish), the principle of energy conservation would only be observed in case the ordinary divergence of the vector $\mathbf{T}^{00}$, $\mathbf{T}^{01}$, $\mathbf{T}^{02}$, $\mathbf{T}^{03}$ is nevertheless zero (in a coordinate system with rectangular coordinates). But this is impossible whenever an object having mass is gathering speed in free anti-radial fall. With its kinetic energy undergoing a change (in the reference frame with rectangular coordinates), the magnitude of $\mathbf{T}^{00}$ is not the same for any value of $\mathbf{x}^0 = \mathbf{t}$.

Therefore, in cases in which objects move in a gravity field in free anti-radial fall, the ordinary divergence of the vector $\mathbf{T}^{00}$, $\mathbf{T}^{01}$, $\mathbf{T}^{02}$, $\mathbf{T}^{03}$ is different from zero (in a reference frame with rectangular coordinates), so that the principle of energy conservation appears to be violated.

(As $\mathbf{T}^{00}$ is an expression of the local energy density in the coordinate system with rectangular coordinates, the magnitude of $\mathbf{T}^{00}$, when referring to the test body, increases with shrinking distance to the central mass, even though there is no relativistic increase in mass of the freely falling test body. In order for the divergence of the vector $\mathbf{T}^{00}, \mathbf{T}^{01}, \mathbf{T}^{02}, \mathbf{T}^{03}$ to be non-zero, it suffices that there is an increase in kinetic energy in the frame of reference used.)

In mathematical terms, the principle of local energy conservation in a closed system can thus be formulated as:

(7a)

$$\int_{space} T^{00}(x^0, x^1, x^2, x^3)\, dx^1 dx^2 dx^3 \;=\; const \;\vee\; \delta_\mu T^{0\mu} \;=\; 0 \;\vee\; \delta_\mu T^{\mu\nu} \;=\; 0$$

4) What the zeroness of the covariant divergence and the possible non-zeroness of the ordinary divergence of the tensor T imply

a) In case it is only the *covariant*, but not the ordinary divergence of **T** which is surely zero (by presupposition) in a rectangular frame of reference (as is the case for Einstein's field equation), energy conservation is still "guaranteed", but under disregard of all gains or losses in kinetic energy which are brought about by "funny coordinates", that is, by gravitation. This also means: In all cases in which objects get accelerated in a gravity field, the ordinary divergence of **T** in a reference frame with rectangular coordinates (for instance, a Cartesian system of coordinates) is not zero (since the kinetic energy picked up during gravitational acceleration by a test body is part of **T**), so that Einstein's field equation appears to allow a violation of the law of conservation of energy. (We will return to this point later on, and we will realize how the principle of energy conservation is saved nevertheless.)

b) Einstein (1952) therefore postulated the following equation as an expression of the principle of conservation of energy in General Relativity (§18, Equation 57 and 57a, p. 151) :

(8)

$$0 \;=\; \frac{\delta T^\alpha_\sigma}{\delta x_\alpha} \;+\; \Gamma_{\sigma\beta}\, T^\alpha_\beta$$

Einstein's equation (8) is equivalent to our Equation (5): Multiplication of both sides of (5) by the metric tensor $\mathbf{g}_{\mu k}$ changes the components of the tensors from contravariant components to mixed components. These can be given different letters thereafter. Explicitly formulating the covariant divergence of the new **T** – which must still vanish – leads to Einstein's equation of energy conservation.

The summand that contains the Christoffel symbol is hence a "correcting" summand which neutralizes, i.e., throws out, the change in **T**, that is the change in energy and momentum (existent in the system of rectangular coordinates), brought about by "funny" spacetime. It cannot be an expression of the change in energy of the gravitational field. If that were the case, that is, if the gravitational field held energy which is exchanged for kinetic energy of falling bodies, the change in that energy of the gravitational field would have to be incorporated in the change in **T**. Then it would be the ordinary divergence of the tensor **T** that would be zero in a rectangular system of coordinates, which would entail that the covariant divergence and hence the right-hand side of Einstein's equation (6) would then be different from zero. This is because the kinetic energy acquired by gravitational acceleration is not given consideration when it comes to determining the covariant divergence of **T** while the change in energy of the gravitational field brought about by the picking up of kinetic energy *would* be given consideration as being part of **T**. But a non-vanishing covariant divergence of **T** would be incompatible with the equality sign in (6).

Pauli (1958/1981) expressed this recognition in the following way (Chapter 61, pp. 175, 176):

"... *in the presence of a gravitational field, the differential equations for the material energy tensor are not*

$\frac{\delta T_i^k}{\delta x^k} \ = \ 0$

as in special relativity, but are of the form

$\frac{\delta T_i^k}{\delta x^k} \ - \ \frac{1}{2} T^{rs} \frac{\delta g_{rs}}{\delta x^4} \ = \ 0$

so that we cannot derive from them the conservation laws

$\int T_i^4 \ dx^1 dx^2 dx^3 \ = \ const$

for a closed system."

5) The not excludable possibility of a non-zero value even of the *covariant* divergence of the tensor T

Despite all this, we cannot exclude apriori that situations might exist in the physical world which would compel us to accept that, contrary to the tacit presupposition made in Equation (1), the covariant divergence of the tensor **T** is non-zero. In such situations (see below for an important example), we would have a choice between two alternatives: We could either discard Einstein's field equation and consider it as being empirically falsified, or we could rather conclude that some energy or mass has entered or left the system which is referred to in Einstein's field equation.

But already at this point, we should be aware of the fact that the turning up of energy from out-of-system is a common postulate in astrophysics: "Dark energy" is commonly conceived of as steadily and ubiquitously "popping up" in empty, expanding space along with new space volume elements (although it is thought of as interacting with other forms of energy by means of gravitation only). This is because of the following: If the tensor **T** in Einstein's field equation is vanishingly small whereas the scalar **lambda** (cosmological constant) is not, Einstein's field equation turns into

(9)

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = - \lambda g^{\mu\nu}$$

The solution of this form of Einstein's field equation is the cosmic variant of the Schwarzschild solution. It presents an expanding space. Apparently, **lambda** times the metric tensor **g** replaces the energy-momentum tensor **T**, and new energy in the form of "dark energy" appears to enter the system all the time and everywhere from outside. We will come back to this point later on, and will make some corrections to this view. Nevertheless, even in its uncorrected form, this widely accepted view on cosmic "dark energy" is helpful in adopting the recognition that energy may pop up that originated at some place outside of the system.

6) The unimportance of potential energy for the local conservation of energy in General Relativity

But what about potential energy? One might think that even the *ordinary* divergence of **T** in a coordinate system with rectangular coordinates (for instance, a Cartesian frame of reference) always turns into zero (even if spacetime is not flat) whenever the potential energy is included in **T**.

But that's not a viable treatment of the situation. Potential energy (in absolute numbers) of a mass element sitting in a force field is simply the difference between the actual energy contents of the force field and the energy contents which the force field would have after the mass element has been brought out of the force field. Hence, potential energy has no mass and thus no weight (and therefore cannot be part of the tensor **T**); only the force field (which contains energy) has.

As an example, consider two spheres. The first one shall be bigger than the second one, and shall be charged with negative electricity. The second, small sphere shall be charged with positive electricity. Imagine the small sphere sits very close to the large sphere to start with. The mass of the system is constituted by the mass of the two spheres plus the mass of the electrostatic field (which has energy). Next, the small sphere shall be brought to a place far away from the big sphere. In order to achieve this goal, work has to be invested, as the attractive electrostatic force between the two spheres has to be overcome. After the small sphere has reached its destination, the energy contents of the total electrostatic field has increased by the amount of work absorbed by the small sphere. So has the *mass* of the electrostatic field. The principle of energy conservation is thus observed without giving consideration to potential energy. More precisely: There is simply no room for ascribing "potential energy" any mass at all.

But in order for any form of energy to be included in the tensor **T**, it *would have to have* mass.

The limited role of potential energy was high-lighted by Heaviside (1893):

"*Potential energy, when regarded merely as expressive of the work that can be done by forces depending upon configuration, does not admit of much argument. It is little more than a mathematical idea, for there is scarcely any physics in it. It explains nothing.*"

He then continues:

"*But in the consideration of physics in general, it is scarcely possible to avoid the idea that potential energy should be capable of localisation equally as well as kinetic.*"

This localization which Heaviside envisages is done in the form of localizing the distribution of the energy of the electrostatic field before and after the small sphere has been moved.

7) The impossibility for the gravitational field to be a field that contains energy in General Relativity

a) But what about the hypothetical energy of the gravitational field? Could its inclusion into **T** guarantee that the ordinary divergence of **T** is always zero in a coordinate system with rectangular coordinates (as is certainly the case in the electrostatic example), and solve our problem? The answer is in the negative (as has been briefly mentioned already):

Even if the ordinary divergence of **T** in a reference frame with rectangular coordinates (for instance, a Cartesian frame of reference) could thereby be turned into zero, it would then be the *covariant* divergence of **T** which would change from zero to non-zero: As mentioned above, the covariant divergence of **T** disregards any gain or loss in kinetic energy which is due to gravitation. That is to say: It is an expression of the principle of conservation of energy, but with disregard of changes in kinetic energy brought about by gravitational acceleration.

In case there were an analogy with the electrostatic field and with the electric force on charged bodies, the hypothetical energy contents of the gravitational field (which then would be part of the tensor **T**) would have to change while an object is accelerated by it. This, in turn, would necessarily lead to a non-zero value of the covariant divergence of **T** in all situations in which an object is gathering or losing speed in a gravity field: The then non-vanishing energy contents of the gravitational field (as part of **T**) would undergo a change, but this change would not be offset by a change in the kinetic energy of the body moving in the gravity field, since the latter change is being disregarded when it comes to forming the covariant divergence of the tensor **T**. (The throwing out of this amount of kinetic energy is brought into effect by means of the second and third summands in Equation 5 or 6, that is, the summands which contains the Christoffel symbol.) But, as we already know, a non-zero value of the covariant divergence would be mathematically incompatible with the presupposition made in Einstein's field equation.

b) As was shown Trupp (2019), the masslessness of the gravitational field is also revealed by the Schwarzschild metric. We recognize that the "masslessness" of a gravitational field in General Relativity is rooted in the mathematical fact that the covariant divergence of each of the two sides of Einstein's field equation is zero. Consequently, this absence is not restricted to the Schwarzschild metric, but applies to any metric.

The idea that the gravitational field contains energy had been criticized also by E. Schrödinger on a different line of thought. As a telltale sign, the local escape velocity (or the local velocity of free fall from afar) is the same in the Schwarzschild metric as it is in Newtonian physics. This could not be the case if the gravitational field held any mass in General Relativity.

The situation has been correctly described by Misner, Thorne, Wheeler (1973), Chapter 20.4 (Why the energy of the gravitational field cannot be localized), p. 467, as follows:

"*Moreover, 'local gravitational energy-momentum' has no weight. It does not curve space. It does not serve as a source term on the righthand side of Einstein's field equations*."

Some pages further (p.546), one reads:

"*Formalistically, to be sure, the gravitational field does not and cannot make any contribution to the source term that stands on the righthand side of Einstein's field equation*."

This recognition is far from being equivocally accepted. Many textbooks (and even some remarks by Einstein) disseminate the opinion according to which the gravitational field has mass and *does* therefore curve spacetime. Those authors simply believe in an analogy with the electric field (although they tend to give the postulated energy of the gravitational field a negative and not a positive sign), without having gone deeper into Einstein's field equation.

8) The apparent dilemma in which General Relativity finds itself, and how to dissolve it

a) General Relativity appears to have landed itself in a dilemma with respect to the law of conservation of energy. By simply saying that it is funny spacetime which is the cause of a gravitational acceleration, the principle of energy conservation is not yet saved completely. Only if "funny coordinates" were merely the result of our free choice (which we have dismissed), there wouldn't exist any need for a supply of energy from a source outside of the object. The increase in kinetic energy would then only be an artefact. But since "funny coordinates" are not the result of our free choice, but of "funny spacetime", the increase in kinetic energy could be considered as being a mere artefact only as long as no forces act on the moving object, for instance, as long as it does not hit a stationary object where it generates heat. As soon as Newton's apple hits the ground where it generates thermal energy, General Relativity appears to find itself in a dilemma, since the covariant divergence of the tensor **T** is no longer zero (contrary to what is presupposed in Einstein's field equation). Energy (in the form of heat) has emerged out of nothingness.

The undeniable non-zeroness of the covariant divergence of the tensor **T** in such a case is made obvious also by the following reflection: When a freely falling test object gathers speed in the gravitational field of a celestial body, the object's (relativistic) mass does not increase. Nor does the mass of the celestial body. But when the test object ultimately hits the surface of the celestial body, thermal energy is generated. Given energy is equal to mass, the total mass of the system "object and celestial body" has suddenly increased (recall that the gravitational field plays no role here, as it does not contain mass), and is larger than it had been a second before the impact. The increase in mass leads to a non-zeroness both of the ordinary divergence (in the rectangular system of coordinates) and of the covariant divergence of the tensor **T**.

b) Being faced with the situation of the covariant divergence of **T** not being zero gives us a choice between two alternatives: We could either say that General Relativity is self-contradictory (see above for the purely mathematical proof of a zero covariant divergence of **T** in Einstein's field equation), or we could conclude that, according to the presupposed zeroness of the covariant divergence of **T** for a strictly closed system in Einstein's field equation, that field equation tells us that the energy which presents itself in the form of thermal energy when Newton's apple hits the ground has come from outside of the system (referred to in Einstein's field equation), and wasn't there prior to the impact of the apple. Moreover, in order for the proximity principle to be valid, that reservoir, though being located outside of the space and the time referred to in Einstein's field equation, must be very close in a spatial direction other than the three spatial directions which are familiar to us.

Given the many tests General Relativity has successfully passed, we should decide in favor of the second alternative.

c) Pauli (1958/1981) made the following statement in this context (Chapter, 54, p. 158):

"*It simply means that energy and momentum can be transmitted from matter to the gravitational field and vice versa (see details cf. §61).*"

However, it cannot be the gravitational field which absorbs or gives off energy, since the energy of the gravitational field would then have to be incorporated in the tensor **T** (which it isn't), and would have to have mass. It must rather be a reservoir outside of the system.

d) To summarize:

– The principle of energy conservation in a completely closed system requires the ordinary divergence of the tensor **T** to be zero in a reference frame with rectangular coordinates (e.g., a Cartesian frame of reference).

– However, in the vicinity of a spherical mass, the ordinary divergence of the tensor **T** in a coordinate system with rectangular coordinates is not zero when applied to a test body in free anti-radial fall. But as long as there is no collision of the falling test body with a stationary object, the *covariant* divergence of the tensor **T** is zero (as required by Einstein's field equation). The gain in kinetic energy may therefore be regarded as being a mere artefact, whereby the principle of energy conservation is saved.

– In case a collision of the freely falling test body occurs in which thermal energy is generated, the covariant divergence of the tensor **T** is no longer zero, as the mass of the system has increased. Then however, given the covariant divergence of the tensor **T** is zero in a closed system as a presupposition in Einstein's field equation, the energy which turned up must have come from outside of the system referred to in Einstein's field equation.

In mathematical terms (**S** – as a function of $\mathbf{x}^0$, $\mathbf{x}^1$, $\mathbf{x}^2$ and $\mathbf{x}^3$ – denotes the flow of energy in Joule per m^2 and per sec as a four-dimensional vector, with zero temporal components and four spatial components $\mathbf{x}^1$, $\mathbf{x}^2$, $\mathbf{x}^3$ and $\mathbf{x}^4$, the first three of which are zero in magnitude):

(10)

$$\nabla_\mu T^{\mu\nu} \neq 0 \ \wedge \ \delta_\mu T^{\mu\nu} \neq 0 \ \rightarrow \ \frac{\delta S_{x^1}}{\delta x^1} + \frac{\delta S_{x^2}}{\delta x^2} + \frac{\delta S_{x^3}}{\delta x^3} + \frac{\delta S_{x^4}}{\delta x^4} = \frac{\delta S_{x^4}}{\delta x^4} \neq 0$$

The above equation constitutes the result of this article in a nutshell.

The $\mathbf{x}^4$-component of **S** is a component in a spatial direction perpendicular to the three familiar ones. Its existence follows (from the non-vanishing of both the covariant and the ordinary divergence of **T**) as a requirement of the law of local conservation of energy, which has incorporated the principle of action by contact.

e) For comparison, in the hypothetical case in which the gravitational field has energy which is being traded for kinetic energy of the falling test body, the following statement would be valid (in mathematical terms):

(11)

$$\nabla_\mu T^{\mu\nu} \neq 0 \ \wedge \ \delta_\mu T^{\mu\nu} = 0 \ \rightarrow \ \frac{\delta S_{x^4}}{\delta x^4} = \mathbf{0} \ \rightarrow \ \nabla_\mu T^{\mu\nu} = 0$$

Since the ordinary divergence of **T** would be zero, there would be no room for an influx of energy from out of the system. But then there would be no reason for the covariant divergence of **T** to be different from zero. The covariant divergence of **T** would rather have to vanish, as is presupposed – for a closed system like the one we are now dealing with (!) – in Einstein's field equation. It thus turns out that (11) is self-contradictory.

9) The world as a brane in higher dimensions

a) Equation (10) introduced a fourth spatial dimension. This was for a reason abstractly expressed by H. Reichenbach [Reichenbach (1957), §44, p. 279] as follows:

"*The principle of action by contact can be satisfied only for a single choice of the dimensionality of the parameter space; that particular parameter space in which it is satisfied is called the coordinate space or 'real space'.*"

In the situations we are considering, real space can only be a space with four spatial dimensions. This does not collide with the fact that the number of spatial dimensions is only three in all other situations. Again it is H. Reichenbach [Reichenbach (1957), §44, p. 283] who explained why:

"*We can conceive special cases in which the space is four-dimensional, yet in which the perceptual experience is not different from that in three-dimensional space. .... Another case would arise if space were four- (or more) dimensional in its smallest elements, but three dimensional as a whole. The situation would correspond to the case of a thin layer of grains of sand which, although each is three-dimensional if taken individually, taken as a whole forms essentially a two-dimensional space. .... In such a world, a macroscopic structure would have only the degrees of freedom of the three dimensions of space, while an atom would have many more degrees of freedom. Sense perceptions in such a world would not be noticeably different from those of our ordinary world; and conversely, it is in principle possible to infer from our ordinary experiences the higher-dimensional character of the microscopic world.*"

Reichenbach thus postulated the possibility of our world being a spatially four- (or more) dimensional "brane" – without using that expression. The term "brane" was used thereafter by authors like L. Randall [Randall (2005), Chapter 3, p. 51]:

"*.... particles and forces can be trapped on lower-dimensional surfaces called branes, even if the universe has many other dimensions to explore.*"

b) It's perfectly possible that the $\mathbf{x}^4$-component of the **S**-vector [appearing in (10)] which brings energy onto the brane does this from one side only. This would be the case if the brane were be a boundary brane (rather than an embedded brane). L. Randall [Randall (2005), Chapter 3, p. 52/53] describes this situation as follows:

"*... perhaps dimensions are not rolled up, but simply terminate within a finite distance. .... The question is: What happens to particles and energy when* *they reach these boundaries? The answer is that they encounter a brane. In a higher-dimensional world, branes would be the boundaries of the full higher-dimensional space, known as the bulk. The bulk spans every dimension, both on and off the brane. The bulk is therefore 'bulky', whereas, in comparison, the brane is flat (in some dimensions), like a pancake. If branes bordered the bulk in some directions, some of the bulk's dimensions would be parallel to the brane, while other dimensions would lead off it. If the brane is the boundary, the dimensions off the brane would extend only to one side*."

If the brane were an embedded one, the $\mathbf{x}^4$-component of the **S**-vector which brings energy onto the brane could do this from either side. It could thus have a positive or a negative sign. In L. Randall's words [Randall (2005), Chapter 3, p. 55]:

"*.... a non-boundary brane would be like a thin slice of bread within the loaf. A non-boundary brane would still be a lower-dimensional object, ... But non-boundary branes would have higher-dimensional bulk space on either side*."

10) What the presupposed zero value of the covariant divergence of the tensor T means for the mass of "cosmic dark energy"

a) It is commonly believed that the expanding universe is characterized by the steady emergence not of space volume alone, but also of energy of space ("dark energy"), which, however, is thought of as being unable to interact with any other form of energy other than by gravitation. It is thereby assumed that any far-away galaxy finds itself, at any time, near the classical escape velocity from an imagined spherical body which is constituted by the mass contained in a sphere whose center is the Milky Way, and whose radius is the distance from the Milky Way to that galaxy. In a matter-dominated universe, the mass stays constant over time while the density of the mass inside the expanding sphere decreases. In a universe dominated by "dark energy", the mass of the expanding sphere is conceived of as increasing while its *density* stays constant.

b) Let us see what the Schwarzschild solution of Einstein's field equation is telling. From the cosmic variant of the Schwarzschild solution is ($\mathbf{r}_s/\mathbf{r}$ is replaced by $\mathbf{H}^2\mathbf{r}^2/\mathbf{c}^2$, with **r** being circumference of a circle with the Milky Way at its center divided by 2 **pi**, **H** being Hubble's constant), we can infer as the escape velocity of galaxies:
(12)

$$\frac{v_{escape}^2}{c^2} = \frac{H^2r^2}{c^2}\,(1 - \frac{H^2r^2}{c^2})^2$$

For a far-away observer who is stationary with respect to the Milky Way,

the local speed **v'** of galaxies rushing past him or her is simply:
(13)

$$\frac{(v'_{escape})^2}{c^2} = \frac{H^2 r^2}{c^2}$$

The speed of the escaping galaxies rushing past him or her is also the speed of expanding space passing by him or her (who is stationary with respect to the Milky Way).

The difference between the two equations (12) and (13) is rooted in the fact that radially oriented, stationary meter sticks held by the distant stationary observer are shortened in the reference frame of the Milky Way; moreover, the ticks of stationary clocks held by the distant observer undergo a dilation. These phenomena are caused by the rushing of space past the distant observer (who is stationary with respect to the Milky Way).

According to (13), the local velocity of galaxies (and of space) which are rushing past the observer approaches **c** if the distant observer is sitting close to the cosmic event horizon (so that his or her distance **r** from the Milky Way is almost **c/H**). For an observer in the Milky Way and his or her rectangular frame of reference, the speed would have reduced to almost zero according to (12), since a stationary meter stick would have shrunk to almost zero, and a local, stationary clock would have come to a standstill [the bracket in Equation (12) would approach zero]. The cosmic event horizon (seen from the Milky Way) thus shows a similar behaviour as the Schwarzschild horizon does: The velocity of free fall (from far) amounts to almost **c** for a stationary observer who sits in front of the horizon.

c) However, if the space between the observer and the Milky Way were filled with energy of space (dark energy), a local velocity **v'=c** of galaxies [as postulated by Equation (13) for **r=c/H**] would be impossible at a position where time dilation and reduction of length goes to infinity – that is, where the local speed of the flow of space (which is responsible for these effects) is almost **c**. The local speed of galaxies passing by would rather have to be lower than **c**, that is, lower than the speed of flowing space at the cosmic event horizon. This is because gravity exerted by the postulated energy of space would try to pull the escaping galaxies back like a stretchable rope between the Milky Way and a distant galaxy would.

We learnt that the gravitational field has no mass. So it should not come as a surprise to find that expanding space (which, in the cosmic version of the Schwarzschild solution, is the equivalent to the gravitational field around a black hole) has no mass either.

d) aa) From this follows: "Dark energy" as a cosmological phenomenon deserves its name only in case parts of an escaping galaxy collides with an object that is not swept away with the flow of space. Then the covariant divergence of the tensor on the right-hand side of Einstein's field equation (which is the sum of **T** and **-lambda g**) differs from zero, and energy turns up (in the form of thermal energy) that has come from a source outside of the system referred to in Einstein's field equation.

bb) Moreover, if space were filled with "dark energy" – which would then have to be included in the tensor **T** – , the *covariant* divergence of the two sides of Einstein's field equation could not be zero: Analogous to a free fall of a test body in the gravity field of a spherical mass, the increase in kinetic energy of an escaping galaxy can be considered as being a mere artefact as long as the galaxy does not collide with an object which is stationary with respect to the Milky Way. The thermal energy which turns up in such a collision leads to a non-zero value not only of the ordinary divergence of the tensor **T** (in the reference frame of the Milky Way with "unfunny" coordinates), but also to a non-zero value of the *covariant* divergence of that tensor. This is the moment for dark energy to step in: The apparent violation of the principle of energy conservation brought about by the emergence of thermal energy can only be avoided by the assumption that the energy has come from a source outside of the system referred to in Einstein's field equation. But if dark energy were filling space and were therefore included in the tensor **T**, it would not be capable of coming to the rescue from *outside* of the system. It would be inside the system already, where it could not be of any value for the saving of the energy principle. Hence, there would be no way of saving the principle of energy conservation from being violated.

cc) In mathematical terms [as an analogue to (10)], the collision of gas cloud (being part of an escaping galaxy) with a slower moving or stationary object in cosmic space is correctly described as follows:
(14)

$$\nabla_\mu(-\lambda g^{\mu\nu} + T^{\mu\nu}) \neq\ 0\ \wedge\ \delta_\mu(-\lambda g^{\mu\nu} + T^{\mu\nu}) \neq\ 0\ \rightarrow\ \frac{\delta S_{x^4}}{\delta x^4} \neq\ 0$$

For a better understanding, one should keep in mind that the *common* view on "dark energy", too, is an attempt to explain things in the universe by postulating that energy is entering the system (referred to in Einstein's field equation) from outside. In other words: The common view, too, regards "dark energy" as originating in an out-of-system reservoir. Otherwise one would have to rate the postulated steady emergence of dark energy between the Milky Way and distant galaxies as a violation of the principle of energy conservation. Different from the common understanding, however, that dark energy does not enter our three-plus-one-dimensional world until collisions take place in which kinetic energy generated by the expansion of the universe is converted into thermal energy. Also different from the common view, the true "dark energy", once having entered the system, is able to interact with any other forms of energy.

11) Further evidence of a out-of-system energy reservoir as a consequence of the covariant divergence of the tensor T being different from zero in special situations

We found: In General Relativity, any non-zero value of the covariant divergence of the tensor **T** (appearing in Einstein's field equation) is proof of a out-of-system reservoir of energy. As is shown Trupp (2022), the existence of a out-of-system reservoir is also postulated in Special Relativity in the context of accelerating or decelerating objects by non-gravitational forces, for instance when it comes to particles in a "target-experiment".

12) Results:

The following results were obtained from an analysis of Einstein's field equation:

– The gravitational field does not contain any energy.

– When Newton's apple hits the ground where kinetic energy (acquired in the earth's gravitational field during the free fall) converts into heat, energy is supplied to the apple and the ground from outside of the system, that is, from outside of the three-plus-one-dimensional world described in Einstein's field equation (with the additional dimension being time).

– The concept of the world as a spatially three-dimensional "brane" in a spatially higher-dimensional "bulk" follows, on the one hand, from the fact that both the covariant and the ordinary divergence of **T** are non-zero in certain situations, and from the law of local conservation of energy (which contains the principle of action by contact) on the other hand.

– Cosmic "dark energy" in space emerges in the three-plus-one-dimensional world whenever the kinetic energy of galaxies generated by the expansion of space is converted into heat during collisions, and not before or after. Without any collisions occurring, dark energy is on "standby" outside of the three-plus-one-dimensional world.

References

[1] Ciufolini, I., Wheeler, J. A., *Gravitation and Inertia*, Princeton University Press, 1995. https://doi.org/10.1515/9780691190198

[2] Einstein, A., The foundation of the general theory of relativity", translated from "Die Grundlage der allgemeinen Relativitätstheorie, *Annalen der Physik*, **354** (1916), 769; in: A. Einstein, H.A. Lorentz, H. Minkowski, *The Principle of Relativity*, Dover Publ., 1952. https://doi.org/10.1002/andp.19163540702

[3] Fleisch, D.,*A Student's Guide to Vectors and Tensors*, Cambridge University Press, 2012. https://doi.org/10.1017/cbo9781139031035

[4] Heaviside, O., A gravitational and electromagnetic analogy, *The Electrician*, Part I, **31** (1893), 281-282.

[5] Misner, C.W.,Thorne, K.S., Wheeler, J.A., *Gravitation*, 1973.

[6] Pauli, W., *Theory of Relativity*, Dover Publishing, 1958/1981.

[7] Randall, L., *Warped Passages – unraveling the mysteries of the universe's hidden dimensions*, 2005.

[8] Reichenbach, H., *The Philosophy of Space and Time*, Dover Publishing, 1957.

[9] Trupp, A., The energy density of the gravitational field, *Physics Essays*, **32** (2019), 484-496. https://doi.org/10.4006/0836-1398-32.4.484

[10] Trupp, A., Interaction between an Accelerated Mass in Straight Motion and a Hidden Energy Reservoir as a Strict Mathematical Consequence of Special Relativity, *Journal of Modern Physics*, **13** (2022), no. 1, 16-33. https://doi.org/10.4236/jmp.2022.131002

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